Introduction to differential equations

Laplace transform

Def

Let $f(t)$, $t \geq 0$, be a piecewise continuous function of exponential order, i.e. $\exists\, M, c \in \mathbb{R}$ s.t.

$$|f(t)| \leq M e^{ct}, \quad 0 \leq t < +\infty.$$

Then the Laplace transform of $f(t)$ is a function $F(s) = \mathcal{L}(f(t))$, given by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Here $\int_0^\infty e^{-st} f(t)\,dt = \lim_{A \to \infty} \int_0^A e^{-st} f(t)\,dt.$

Example

1) $f(t) \equiv 1$,

$$\mathcal{L}(f(t)) = F(s) = \lim_{A \to +\infty} \int_0^A e^{-st} \cdot 1\,dt = \lim_{A \to \infty} \frac{1 - e^{-st}}{s} =$$

$$= \begin{cases} \frac{1}{s}, & s > 0 \\ \infty, & s \leq 0 \end{cases}$$

2) $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \lim_{A \to \infty} \int_0^A e^{(a-s)t}\, dt =$$

$$= \lim_{A \to \infty} \frac{e^{(a-s)A} - 1}{a - s} = \begin{cases} \frac{1}{s-a}, & s > a \\ \infty, & s \le a \end{cases}$$

3) $f(t) = \cos \omega t$ or $f(t) = \sin \omega t$

$$\mathcal{L}(\cos \omega t) = \int_0^\infty e^{-st} \cos \omega t\, dt$$

$$\mathcal{L}(\sin \omega t) = \int_0^\infty e^{-st} \sin \omega t\, dt$$

$$\mathcal{L}(\cos \omega t) + i\,\mathcal{L}(\sin \omega t) = \int_0^\infty e^{-st} \cdot e^{i\omega t}\, dt =$$

$$= \lim_{A \to \infty} \int_0^A e^{(i\omega - s)t}\, dt = \lim_{A \to \infty} \frac{e^{(i\omega - s)A} - 1}{i\omega - s} =$$

$$= \begin{cases} \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2}, & s > 0 \\ \text{not defined}, & s \le 0 \end{cases}$$

Therefore

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \quad s > 0$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}, \quad s > 0.$$

Remark

$\mathcal{L}$ is a linear operator!

$$\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$$

Lemma 1

If $|f(t)| \leq M e^{ct}$ then $F(s)$ exists for $s > c$.

Proof

$$\left| \int_0^A e^{-st} f(t)\, dt \right| \leq \int_0^A e^{-st} |f(t)|\, dt \leq$$

$$\leq \int_0^A M e^{ct} \cdot e^{-st}\, dt = M \int_0^A e^{(c-s)t}\, dt =$$

$$= \frac{M}{c-s} \left(e^{(c-s)A} - 1 \right) \leq \frac{M}{s-c} \quad \text{if } s > c$$

$\square$

Lemma 2

Suppose $F(s) = \mathcal{L}(f(t))$. Then

$$\mathcal{L}(f'(t)) = sF(s) - f(0).$$

Proof

$$\mathcal{L}(f'(t)) = \lim_{A \to \infty} \int_0^A e^{-st} f'(t)\, dt =$$

$$= \lim_{A \to \infty} \left(e^{-st} f(t) \Big|_0^A + s \cdot \int_0^A e^{-st} f(t)\, dt \right) =$$

$$= -f(0) + \lim_{A \to \infty} s \int_0^A e^{-st} f(t)\, dt = -f(0) + s \cdot F(s)$$

$\square$

Lemma 3

If $F(s) = \mathcal{L}(f(t))$ then

$$\mathcal{L}(f''(t)) = s^2 F(s) - s \cdot f(0) - f'(0)$$

Proof

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) =$$

$$= s\,(sF(s) - f(0)) - f'(0) =$$

$$= s^2 F(s) - s \cdot f(0) - f'(0) \qquad \square$$

How to use the Laplace transform to solve differential equations?

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'$$

Apply the Laplace transform to both parts:

$$\mathcal{L}(ay'' + by' + cy) = \mathcal{L}(f(t))$$

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = F(s), \quad \text{where } F(s) = \mathcal{L}(f(t))$$

Denote $Y(s) = \mathcal{L}(y(t))$, then

$$\mathcal{L}(y') = sY(s) - y_0$$

$$\mathcal{L}(y'') = s^2 Y(s) - sy_0 - y_0', \text{ so}$$

$$a(s^2 Y(s) - sy_0 - y_0') + b(sY(s) - y_0) + cY(s) = F(s),$$

$$Y(s) = \frac{(as+b)y_0}{as^2+bs+c} + \frac{ay_0'}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$$

If we can find $y(t)$ such that $\mathcal{L}(y(t)) = Y(s)$ then $y(t)$ is the solution of the initial value problem.

Example

1) $y' - y = 1 \quad , \quad y(0) = 0$

$$\mathcal{L}(y' - y) = \mathcal{L}(1)$$

$\mathcal{L}(y') = sY(s) - y(0)$, therefore

$$sY(s) - Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s-1)}$$

$$Y(s) = \frac{s-(s-1)}{s(s-1)} = \underbrace{\frac{1}{s-1}}_{\mathcal{L}(e^t)} - \underbrace{\frac{1}{s}}_{\mathcal{L}(1)} = \mathcal{L}(e^t - 1),$$

therefore $y(t) = e^t - 1$ is the solution of the initial equation.

Post's inversion formula

If $F(s) = \mathcal{L}(f(t))$ then

$$f(t) = \mathcal{L}^{-1}(F(s)) = \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right),$$

$t > 0$.

Example

2)
$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1, \; y'(0) = 0$$

$$s^2 Y(s) - s - 3(sY(s) - 1) + 2Y(s) = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s-3)(s^2-3s+2)} + \frac{s-3}{s^2-3s+2} =$$

$$= \frac{1}{(s-1)(s-2)(s-3)} + \frac{s-3}{(s-1)(s-2)} =$$

$$= \frac{5}{2} \cdot \frac{1}{s-1} - \frac{2}{s-2} + \frac{1}{2}\,\frac{1}{s-3} =$$

$$= \mathcal{L}\left(\frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}\right) \Rightarrow$$

$$y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

Laplace transform of some other functions:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0$$

$$\mathcal{L}\left(\frac{t^n}{n!}e^{-\alpha t}\right) = \frac{1}{(s+\alpha)^{n+1}}, \quad s > -\alpha$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\mathcal{L}(\sinh(\alpha t)) = \frac{\alpha}{s^2-\alpha^2}, \quad s > |\alpha|$$

$$\mathcal{L}(\cosh(\alpha t)) = \frac{s}{s^2-\alpha^2}, \quad s > |\alpha|$$

$$\mathcal{L}\left(\ln\left(\frac{t}{t_0}\right)\right) = -\frac{t_0}{s}\left(\ln(t_0 s) + \gamma\right), \quad \gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k} - \ln n\right) \approx$$

$$\approx 0.57721566\ldots$$