Series solutions

$$L(y) = y'' + p(t)\,y' + q(t)\,y = 0$$

Suppose that $p(t)$, $q(t)$ are polynomials or analytic functions in t. Let us try to find a solution $y(t)$ in the form

$$y(t) = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{n=0}^{\infty} a_n t^n$$

Example

$$y'' - 2t y' - 2y = 0$$

$$y(t) = a_0 + a_1 t + \ldots = \sum_{n=0}^{\infty} a_n t^n$$

$$y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \ldots = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

$$y''(t) = 2a_2 + 6a_3 t + \ldots = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2},$$

therefore

$$L(y) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 2t \sum_{n=0}^{\infty} n a_n t^{n-1} - 2 \sum_{n=0}^{\infty} a_n t^n =$$

$$= \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 2 \sum_{n=0}^{\infty} n a_n t^n - 2 \sum_{n=0}^{\infty} a_n t^n = 0$$

Since $\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} t^n =$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n, \text{ so}$$

$$L(y) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - 2\sum_{n=0}^{\infty} n a_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$(n+2)(n+1)a_{n+2} - 2n a_n - 2a_n = 0$$

$$a_{n+2} = \frac{2(n+1)a_n}{(n+1)(n+2)} = \frac{2a_n}{n+2}$$

Fix any a_0, a_1. Then $y(0) = a_0$, $y'(0) = a_1$. We have

$$a_2 = a_0, \quad a_4 = \frac{2a_2}{4}, \quad a_4 = \frac{2a_4}{6}, \ldots$$

For example, if $y(0) = 1$, $y'(0) = 0$ then

$$a_{2n} = \frac{1}{2\cdot 3\cdot \ldots \cdot n} = \frac{1}{n!}, \text{ so}$$

$$y_1(t) = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \ldots = e^{t^2}.$$

If $y(0) = 0$, $y'(0) = 1$, then

$$a_3 = \frac{2a_1}{3} = \frac{2}{3}, \quad a_5 = \frac{2a_3}{5} = \frac{2}{3}\cdot\frac{2}{5} = \frac{4}{15}, \ldots$$

$$a_{2n+1} = \frac{2^n}{3\cdot 5\cdot \ldots \cdot (2n+1)},$$

$$y_2(t) = t + \frac{2t^3}{3} + \frac{2^2 t^5}{3\cdot 5} + \ldots = \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{3\cdot 5\cdot \ldots \cdot (2n+1)}.$$

Def

A function $f: (\alpha, \beta) \to \mathbb{R}$ is analytic if

$\forall\, t_0 \in (\alpha, \beta)$ $\exists\, (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\forall\, t_0 \in (t_0 - \varepsilon, t_0 + \varepsilon)$

$$f(*) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \text{ - convergent series.}$$

Remark

In this case $a_n = \dfrac{f^{(n)}(t_0)}{n!}$, $f^{(n)}(t_0) = \dfrac{d^n f(t)}{dt^n}\Big|$

Def

An expression

$$(*) \qquad f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \quad \text{is a power series.}$$

Def

Interval of convergence:

$\exists\, \rho \geq 0$ s.t. the series $(*)$ converges for $|t - t_0| < \rho$ and diverges for $|t - t_0| > \rho$.

Cauchy ratio test:

Suppose $\exists \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda$. Then $\rho = \frac{1}{\lambda}$

(i.e. $(*)$ converges if $|t - t_0| < \frac{1}{\lambda}$, and

diverges if $|t - t_0| > \frac{1}{\lambda}$).

Hadamard formula

$$\rho^{-1} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$$

Example

What is the radius of convergence of

$$f(t) = 1 + t + t^2 + t^3 + \dots \quad ?$$

$a_n = 1 \ \forall n$. <u>Answer: $\rho = 1$</u>, so interval of convergence is $(-1, 1)$

$$f(t) = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \frac{t^3}{2^3} + \dots,$$

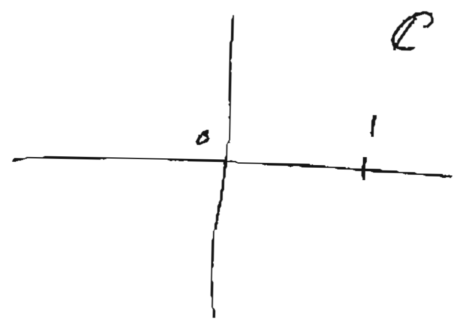
$a_n = 2^{-n}$. <u>Answer: $\rho = 2$</u>, $(-2, 2)$.

Theorem

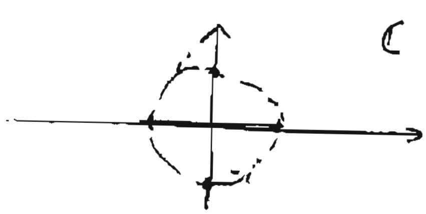
Consider $f(t) = \sum_{n=0}^{\infty} a_n t^n$ as a function of <u>complex</u> variable t. Let z_0 be the closest to 0 point where f or one of its derivatives does not exists. Then $\rho = |z_0|$ is the radius of convergence of the series $f(t)$.

Example

1) $f(t) = 1 + t + t^2 + \dots = \dfrac{1}{1-t}$



2) $f(t) = 1 + t^2 + t^4 - t^6 + t^8 - \dots = \dfrac{1}{1+t^2}$



$1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots = ?$

$1?$ $-1?$ $\quad 1/2\,!$

– $f(t)$, $g(t)$ are analytic, then $f(t)\cdot g(t)$ is also analytic, and if ρ_f, ρ_g are radii of convergence, then $\rho_{f\cdot g} \geq \min(\rho_f, \rho_g)$.

Theorem

$$y'' + p(t)y' + q(t)y = 0$$

If $p(t)$, $q(t)$ are analytic at $t = t_0$ ~~then~~ and converge on $|t - t_0| < \rho$ then every solution $y(t)$ is also analytic and converges on $|t - t_0| < \rho$

Coeff. $a_2, a_3, \dots$, in $y(t) = \sum_{n=0}^{\infty} a_n t^n$ can be determined from coeff. of $p(t)$ and $q(t)$ by plugging $y(t)$ into the equation.

Example

$$y'' + t^2 y' + 2ty = 0, \quad y(0) = 1, \; y'(0) = 0.$$

$$L(y) = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + t^2 \sum_{n=0}^{\infty} n a_n t^{n-1} + 2t \sum_{n=0}^{\infty} a_n t^n =$$

$$= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} n a_n t^{n+1} + 2\sum_{n=0}^{\infty} a_n t^{n+1} =$$

$$= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} (n+2)a_n t^{n+1}$$

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2)\, a_{n+3}\, t^{n+1},$$

$$L(y) = 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2)a_{n+3} t^{n+1} + \sum_{n=0}^{\infty} (n+2)\, a_n t^{n+1} = 0$$

$$a_2 = 0, \quad (n+3)(n+2)a_{n+3} + (n+2)\, a_n = 0, \quad n = 0, 1, \ldots,$$

$$a_2 = 0, \quad a_{n+3} = -\frac{a_n}{n+3}$$

Set $a_0 = 1$, $a_1 = 0$ (this is our IVP),

all $a_5, a_8, a_{11}, \ldots$ are 0, and

$a_4, a_7, a_{10}, \ldots,$ are 0.

$$a_3 = -\frac{a_3}{3} = -\frac{1}{3}, \quad a_6 = -\frac{a_3}{6} = \frac{1}{3\cdot 6}, \quad a_9 = -\frac{a_6}{9} = -\frac{1}{3\cdot 6\cdot 9},$$

$$a_{3n} = \frac{(-1)^n}{3\cdot 6 \cdots 3n} = \frac{(-1)^n}{3^n \cdot n!}$$

$$y(t) = 1 - \frac{t^3}{3} + \frac{t^6}{3\cdot 6} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n}}{3^n\, n!}$$

– converges for all t.