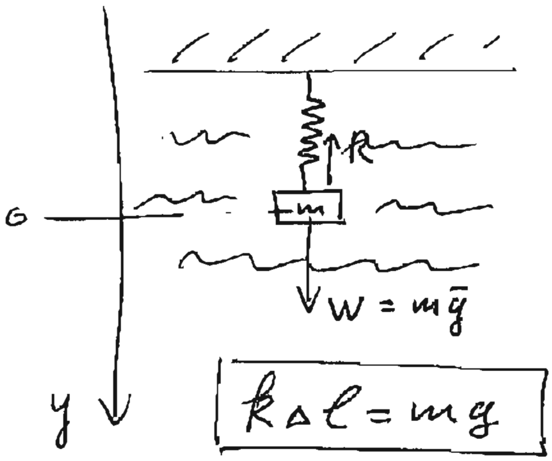
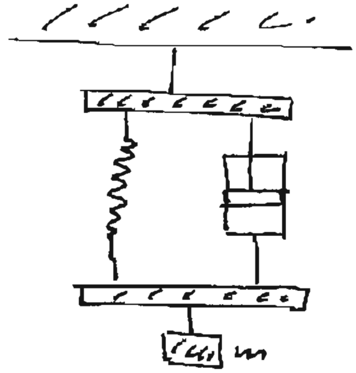
Introduction to differential equations

Oct. 24, 2008

Damped and/or forced oscillations



$$\boxed{k\Delta l = mg}$$



W - gravitation mg

R - restoring force of the spring $-(\Delta l + y)k$

D - force of the damping („drag force") $-c\,\frac{dy}{dt}$

F - external force (given explicitely)

Newton's second law of motion:

$$m\,\frac{d^2y}{dt^2} = W + R + D + F =$$
$$= mg - k(\Delta l + y) - c\,\frac{dy}{dt} + F(t) =$$
$$= -ky - c\,\frac{dy}{dt} + F(t),$$

$$my'' + ky + cy' = F(t)$$

$$m, k, c > 0$$

Free vibrations: (no damping, no external force)

$$my'' + ky = 0 \quad , \quad \frac{k}{m} = \omega_0^2$$

$$y'' + \omega_0^2 y = 0$$

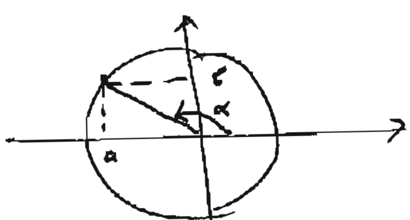
Characteristic function: $r^2 + \omega_0^2 r = 0$

$$r_{1,2} = \pm \omega_0 i$$

$$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t =$$

$$= \sqrt{c_1^2 + c_2^2} \Big(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{a} \cos \omega_0 t + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{b} \sin \omega_0 t \Big)$$

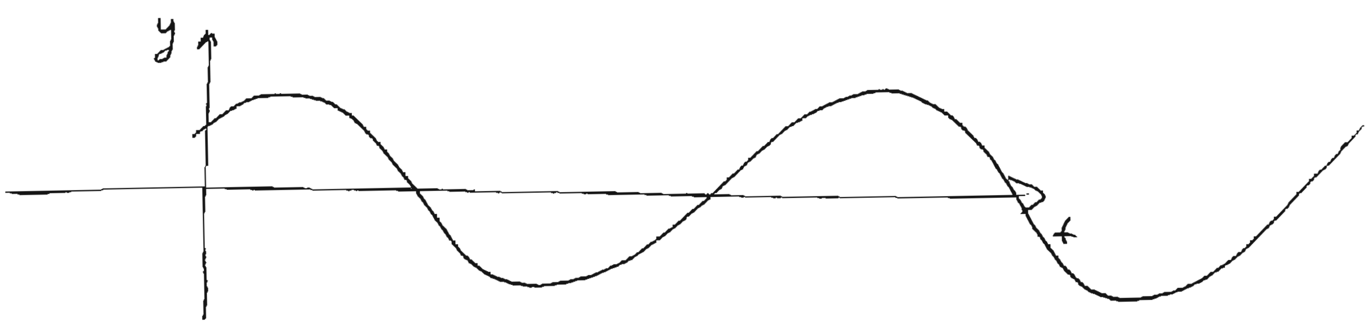
$a^2 + b^2 = 1 \Rightarrow \exists\, \alpha$ s.t. $a = \cos \alpha$, $b = \cancel{\cos\alpha}\ \sin \alpha$



Therefore

$$y(t) = R\,(\cos\alpha \cos \omega_0 t + \sin\alpha \sin \omega_0 t) =$$

$$= R \cos(\omega_0 t - \alpha) \qquad \left(R = \sqrt{c_1^2 + c_2^2}\right)$$



Damped free vibrations: (no external force)

$$my'' + cy' + ky = 0$$

$$mr^2 + cr + k = 0$$

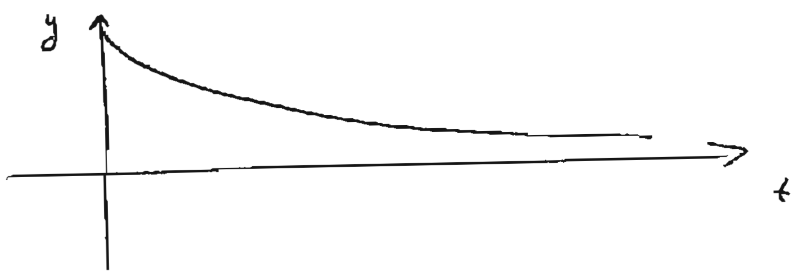
$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m}, \quad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}$$

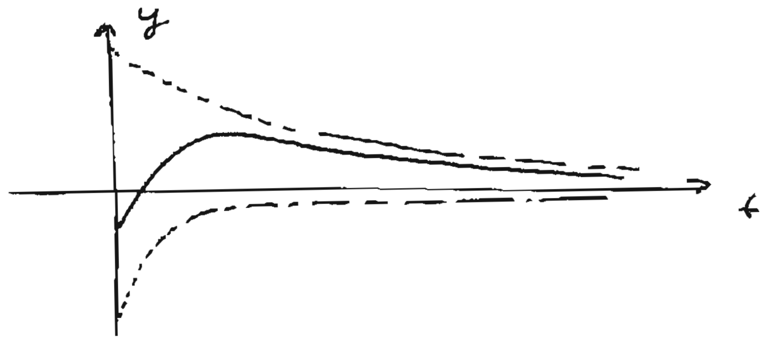
Case 1 (overdamped)

$c^2 - 4km > 0$,

r_1, r_2 are real and different, $r_1, r_2 < 0$,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$





Case 2 (critically damped)

$$c^2 - 4km = 0$$

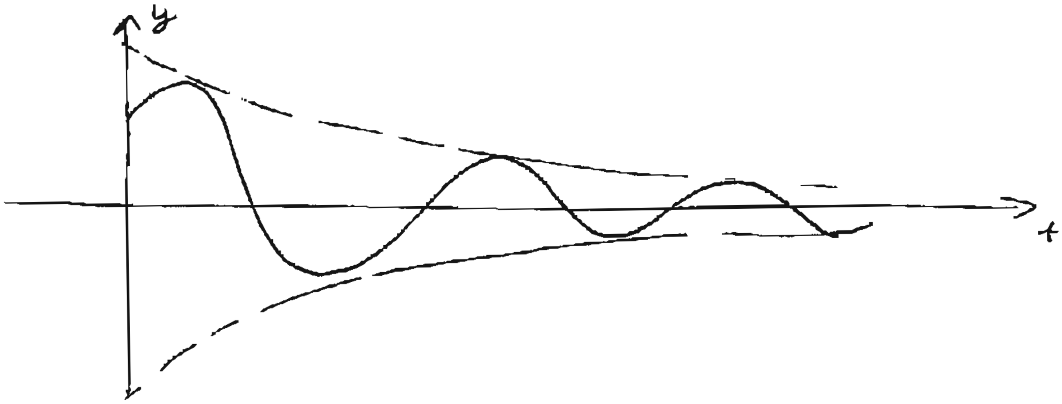
$$y(t) = (c_1 t + c_2) e^{-\frac{c}{2m} t}$$

Case 3

$$c^2 - 4km < 0$$

$$y(t) = e^{-\frac{ct}{2m}} \left(C_1 \cos \mu t + C_2 \sin \mu t \right),$$

$$y(t) = R e^{-\frac{ct}{2m}} \cos(\mu t - \alpha) \qquad \mu = \frac{\sqrt{4km - c^2}}{2m}.$$



Dumped forced vibrations

$$m y'' + c y' + k y = F_0 \cos \omega t$$

A partial solution can be found in a form

$$\psi(t) = A_0 \cos \omega t + A_1 \sin \omega t$$

$$\psi'(t) = -A_0 \omega \sin \omega t + A_1 \omega \cos \omega t$$

$$\psi''(t) = -A_0 \omega^2 \cos \omega t - A_1 \omega^2 \sin \omega t = -\omega^2 \psi(t)$$

$$m\left(-A_0 \omega^2 \cos \omega t - A_1 \omega^2 \sin \omega t\right) +$$
$$+ c\left(-A_0 \omega \sin \omega t + A_1 \omega \cos \omega t\right) + k\left(A_0 \cos \omega t + A_1 \sin \omega t\right) =$$
$$= F_0 \cos \omega t$$

$$\begin{cases} -mA_0\omega^2 + cA_1\omega + kA_0 = F_0 \\ -mA_1\omega^2 + c(-A_0\omega) + kA_1 = 0 \end{cases}$$

$$\begin{cases} (k - m\omega^2)A_0 + c\omega A_1 = F_0 \\ -c\omega A_0 + (k - \cancel{A_1}\, m\omega^2)A_1 = 0 \end{cases} \qquad \begin{array}{|l} k - m\omega^2 \\ c\omega \end{array}$$

$$(k - m\omega^2)^2 A_0 + c^2\omega^2 A_0 = F_0(k - m\omega^2)$$

$$A_0 = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + c^2\omega^2}$$

$$A_1 = \frac{c\omega F_0}{(k - m\omega^2)^2 + c^2\omega^2}$$

$$\psi(t) = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2}\left[(k - m\omega^2)\cos\omega t + c\omega \sin\omega t\right] =$$

$$= \frac{F_0}{((k - m\omega^2)^2 + c^2\omega^2)^{1/2}} \cos(\omega t - \alpha)$$

$y(t) =$ solution of a homogeneous equation $+ \psi(t)$

$\searrow 0$ as $t \to +\infty$

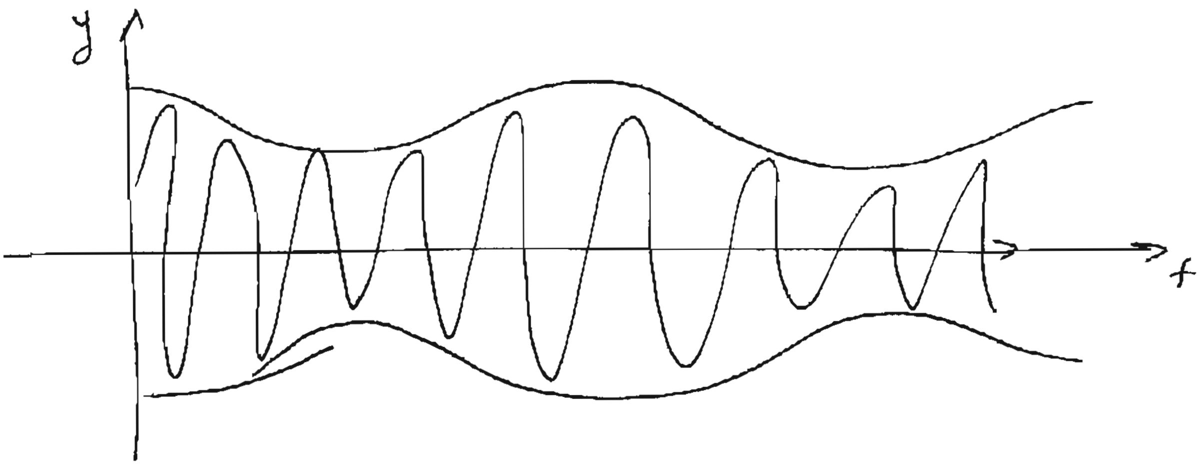
Forced free oscillations

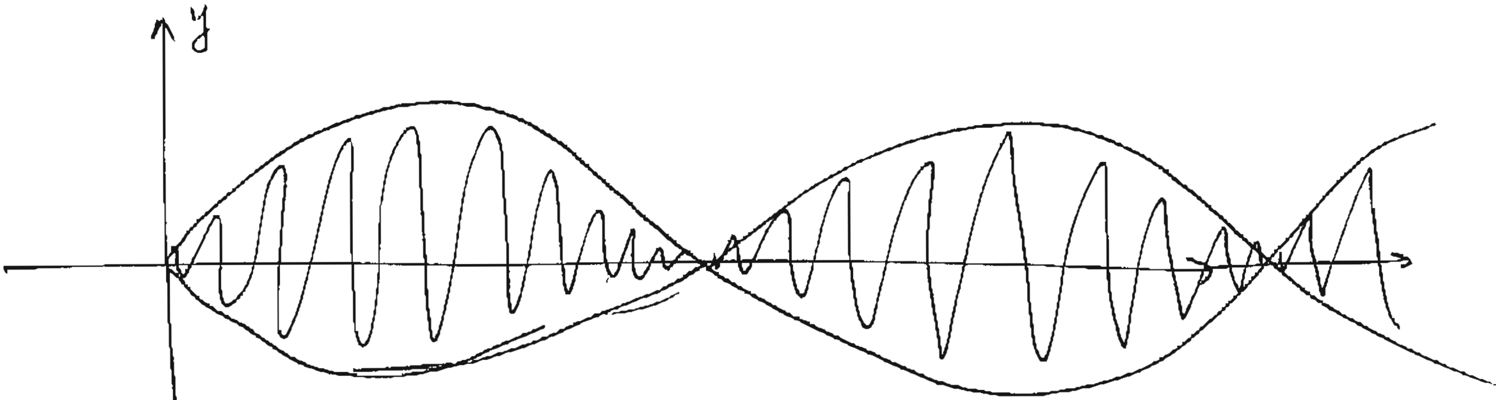
$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t \quad , \quad \omega_0^2 = \frac{k}{m} > 0$$

If $\omega \neq \omega_0$ then

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{k - m\omega^2} \cos \omega t =$$

$$= R \cos(\omega_0 t - \alpha) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$





$\left(\text{if } R = \frac{F_0}{m(\omega_0^2 - \omega^2)} \right)$

(simple beats)

If $\omega = \omega_0$ then

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t$$

Complexify:

$$y'' + \omega_0^2 y = \frac{F_0}{m} e^{i\omega_0 t} \quad , \quad \cos \omega_0 t = \mathrm{Re}\, e^{i\omega_0 t}$$

Partial solution

$$\psi(t) = A t e^{i\omega_0 t}$$

$$\psi'(t) = A e^{i\omega_0 t} + A t i \omega_0 e^{i\omega_0 t}$$

$$\psi''(t) = A i \omega_0 e^{i\omega_0 t} + A i \omega_0 e^{i\omega_0 t} - A t \omega_0^2 e^{i\omega_0 t}$$

$$2 A i \omega_0 - A t \omega_0^2 + A t \omega_0^2 = \frac{F_0}{m}$$

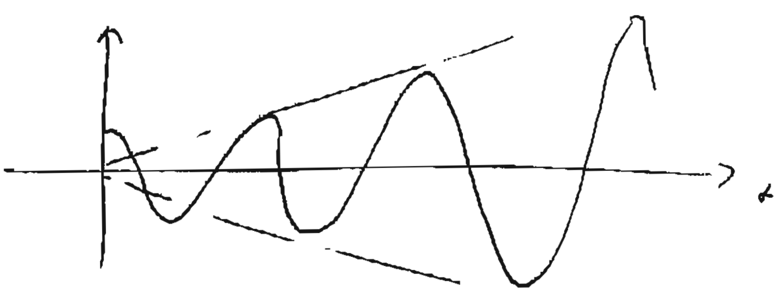
$$A = \frac{-F_0 i}{2 \omega_0 m}$$

$$\psi(t) = -\frac{F_0 i}{2 \omega_0 m} t e^{i\omega_0 t} =$$

$$= -\frac{F_0 t}{2\omega_0 m} \left(-\sin \omega_0 t + i \cos \omega_0 t \right),$$

$$\mathrm{Re}\, \psi(t) = \frac{F_0 t}{2 \omega_0 m} \sin \omega_0 t,$$

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{2\omega_0 m} t \sin \omega_0 t$$



<u>Resonance</u> !