Introduction to differential equations

Oct. 22, 2006

Non-homogeneous second-order linear differential equations with constant coefficients.

$$a y'' + b y' + c y = g(t) \quad , a, b, c \in \mathbb{R}.$$

Suppose that $g(t)$ is a polynomial.

$$L(y) \equiv a y'' + b y' + c y = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

Let us try to find a solution in the form

$$\psi(t) = A_0 + A_1 t + \dots + A_n t^n$$

$$L(\psi) = a\psi'' + b\psi' + c\psi = a\left[2A_2 + \dots + n(n-1)A_n t^{n-2}\right] +$$

$$+ b\left[A_1 + \dots + nA_n t^{n-1}\right] + c\left[A_0 + A_1 t + \dots + A_n t^n\right] =$$

$$= cA_n t^n + (cA_{n-1} + bnA_n)t^{n-1} +$$

$$+ (cA_{n-2} + b(n-1)A_{n-1} + n(n-1)A_n \cdot a)t^{n-2} +$$

$$\dots + (2aA_2 + bA_1 + cA_0) =$$

$$\cancel{= A_0 + A_1 t + \dots + A_n t^n}$$

$$= a_0 + a_1 t + \dots + a_n t^n$$

From here we have

$$A_n = \frac{a_n}{c} \quad (\underline{\text{if } c \neq 0}),$$

$$c A_{n-1} + b n A_n = a_{n-1}, \text{ so}$$

$$A_{n-1} = \frac{1}{c}\left(a_{n-1} - b_n \cdot \frac{a_n}{c}\right)$$

— — — .

If $c = 0$ then let us try to find a partial solution as $\psi(t) = t\,(A_0 + A_1 t + \dots + A_n t^n)$

So we take

$$\psi(t) = \begin{cases} A_0 + A_1 t + \dots + A_n t^n, & c \neq 0 \\ t\,(A_0 + \dots + A_n t^n), & c = 0, b \neq 0 \\ t^2 (A_0 + \dots + A_n t^n), & c = b = 0. \end{cases}$$

<u>Example</u>

$$y'' - y' = t \qquad r^2 - r = 0$$
$$r = \{0, 1\}$$

$$\psi = t\,(A_0 + A_1 t)$$

$$2A_1 - (A_0 + 2A_1 t) = t$$

$$\begin{cases} 2A_1 - A_0 = 0 \\ -2A_1 = 1 \end{cases} \Rightarrow \begin{cases} A_1 = -\frac{1}{2} \\ A_0 = -1 \end{cases}$$

$\psi(t) = t\,(-1 - \frac{1}{2}t)$ is a partial solution

$y(t) = C_1 e^t + C_2 + (-1 - \frac{1}{2}t)\,t$ is a general solution.

If $g(t) = (a_0 + a_1 t + \ldots + a_n t^n)\, e^{\alpha t}$ then

try $y(t) = e^{\alpha t} \cdot v(t)$,

$$y' = e^{\alpha t}(v' + \alpha v)$$

$$y'' = e^{\alpha t}(v'' + 2\alpha v' + \alpha^2 v), \text{ so}$$

$$L(y) = e^{\alpha t}\left(\alpha v'' + (2a\alpha + b)v' + (a\alpha^2 + b\alpha + c)v\right),$$

so

$$\alpha v'' + (2a\alpha + b)v' + \underbrace{(a\alpha^2 + b\alpha + c)}_{\text{charact. function.}} v = a_0 + a_1 t + \ldots + a_n t$$

Therefore, if α is <u>not</u> a root of characteristic function, then

$$\psi(t) = e^{\alpha t}(A_0 + \ldots + A_n t),$$

if α is a root (single!) of a charact. function then

$$\psi(t) = e^{\alpha t} \cdot t\,(A_0 + \ldots + A_n t);$$

if α is a double root of characteristic polynomial then

$$\psi(t) = e^{\alpha t} \cdot t^2 (A_0 + \ldots + A_n t).$$

Example

$$y'' - 2y' - 3y = e^{4t}$$

$$r^2 - 2r - 3 = 0$$

$$D = 4 + 12 = 16$$

$$r = \frac{2 \pm 4}{2} = \{3, -1\}$$

$$\psi(t) = e^{4t} \cdot A_0$$

$$16 A_0 e^{4t} - 2 \cdot 4 A_0 e^{4t} - 3 A_0 e^{4t} = e^{4t}$$

$$16 A_0 - 8 A_0 - 3 A_0 = 1$$

$$5 A_0 = 1$$

$A_0 = \frac{1}{5}$, so $\psi(t) = \frac{1}{5} e^{4t}$ is a partial solution,

so general solution is

$$y(t) = C_1 e^{3t} + C_2 e^{-t} + \frac{1}{5} e^{4t}$$

Remark

$$L(y) = g_1(t) + g_2(t)$$

If $\psi_1(t)$ is a partial solution of the equation

$$L(y) = g_1(t),$$

and $\psi_2(t)$ — " — $L(y) = g_2(t)$, then

$\psi_1(t) + \psi_2(t)$ is a partial solution of the initial equation.

Lemma

If $y(t) = u(t) + i v(t)$ is a complex-valued solution of $L(y) = a y'' + b y' + c y = g_1(t) + i g_2(t)$,

$$\underline{a, b, c \in \mathbb{R}},$$

then $L(u) = g_1(t)$, $L(v) = g_2(t)$.

This allows to solve the equations of the form

$$a y'' + b y' + c y = e^{\alpha t}(a_0 + a_1 t + \dots + a_n t^n) \sin \beta t$$

(or $\cos \beta t$).

Indeed, $e^{\alpha t} \cos \beta t = \operatorname{Re} e^{(\alpha + i\beta) t}$

$$e^{\alpha t} \sin \beta t = \operatorname{Im} e^{(\alpha + i\beta) t}$$

Example

$$y'' + y = \sin 2t$$

$$r^2 + 1 = 0$$

$$r = \pm i, \quad \sin 2t = \operatorname{Im} e^{2it}$$

$$y'' + y = e^{2it}$$

$$\psi(t) = e^{2it} \cdot A_0$$

$$-4 A_0 e^{2it} + A_0 e^{2it} = e^{2it}$$

$$\underline{A_0 = -\tfrac{1}{3}}, \text{ so } \psi(t) = -\tfrac{1}{3} e^{2it},$$

$$\operatorname{Im} \psi(t) = -\tfrac{1}{3} \sin 2t$$

General solution:

$$y(t) = C_1 \cos t + C_2 \sin t - \frac{1}{3} \sin 2t$$

Example

$$y'' + 2y' + y = 1 + e^{t} + e^{2t}$$

$$r^2 + 2r + 1 = 0$$

$$\underline{r = -1}$$

$$y'' + 2y' + y = 1 \quad , \quad \psi(t) = 1$$

$$y'' + 2y' + y = e^{t} \quad , \quad \psi(t) = A_0 e^{t}$$

$$(A_0 + 2A_0 + A_0) e^{t} = e^{t}$$

$$A_0 = \frac{1}{4} \quad , \quad \psi(t) = \frac{1}{4} e^{t}$$

$$y'' + 2y' + y = e^{2t} \quad , \quad \psi(t) = A_0 e^{2t}$$

$$4A_0 e^{2t} + 4A_0 e^{2t} + A_0 e^{2t} = e^{2t}$$

$$9A_0 = 1 \quad , \quad \psi(t) = \frac{1}{9} e^{2t}$$

General solution of the initial equation:

$$y(t) = (C_1 + C_2 t) e^{-t} + 1 + \frac{1}{4} e^{t} + \frac{1}{9} e^{2t}$$

Example

$$y'' - 2y' + 5y = 2\cos^2 t$$

$$r^2 - 2r + 5 = 0$$

$$D = 4 - 20 = -16$$

$$r = \frac{2 \pm 4i}{2} = \{1+2i, 1-2i\}$$

$$2\cos^2 t = 1 + \cos 2t$$

$$y'' - 2y' + 5y = 1, \quad \psi = \frac{1}{5}$$

$$y'' - 2y' + 5y = \cos 2t = \operatorname{Re} e^{2it}$$

$$y'' - 2y' + 5y = e^{2it}, \quad \psi(t) = A_0 e^{2it}$$

$$-4A_0 e^{2it} - 4i A_0 e^{2it} + 5A_0 e^{2it} = e^{2it}$$

$$(1 - 4i) A_0 = 1$$

$$A_0 = \frac{1}{1-4i} = \frac{1+4i}{1+16} = \frac{1+4i}{17}, \quad \psi(t) = \frac{1+4i}{17} e^{2it}$$

$$\operatorname{Re} \psi(t) = \frac{1}{17}\cos 2t - \frac{4}{17}\sin 2t$$

General solution:

$$y(t) = C_1 e^t \cos 2t + C_2 e^t \sin 2t + \frac{1}{5} +$$

$$+ \frac{1}{17}\cos 2t - \frac{4}{17}\sin 2t$$