Introduction to differential equations Oct. 20, 2002

Second order linear differential equations.

The non-homogeneous equation.

$$L(y) = \underline{y'' + p(t)\,y' + q(t)\,y = g(t)}, \qquad (*)$$

$p(t), q(t), g(t)$ are continuous on (α, β).

Thm

Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of

$$y'' + p(t)\,y' + q(t)\,y = 0,$$

and $\psi(t)$ be a solution of

$$y'' + p(t)\,y' + q(t)\,y = g(t) \qquad (*)$$

Then a general solution of $(*)$ has the form

$$C_1 y_1(t) + C_2 y_2(t) + \psi(t).$$

Proof

We need to show that any function of this form is a solution of $(*)$, and that any solution of $(*)$ has this form.

$L(C_1 y_1(t) + C_2 y_2(t) + \psi(t)) =$

$= C_1 L(y_1(t)) + C_2 L(y_2(t)) + L(\psi(t)) =$

$= C_1 \cdot 0 + C_2 \cdot 0 + g(t) = g(t)$, so

$C_1 y_1(t) + C_2 y_2(t) + \psi(t)$ is a solution of $(*)$.

Assume that $y(t)$ is a solution of $(*)$, then $L(y-\psi) = L(y) - L(\psi) = g(t) - g(t) \equiv 0$, so $y-\psi$ is a solution of a homogeneous equation

$$y'' + p(t) y' + q(t) y = 0,$$

so $y(t) - \psi(t) = C_1 y_1(t) + C_2 y_2(t)$,

and $\underline{y(t) = C y_1 + C_2 y_2 + \psi}$ $\square$

<u>Example</u>

Suppose that $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are solutions of $(*)$, and $W[\psi_1 - \psi_3, \psi_1 - \psi_2] \neq 0$. Find a general solution

<u>Solution</u>: $L(\psi_1 - \psi_3) = 0 = L(\psi_1 - \psi_2)$, so $\psi_1 - \psi_3$ and $\psi_1 - \psi_2$ are (linearly indep.!) solutions of $(*)$. Therefore a general solution has the form $\psi_1 + C_1(\psi_1 - \psi_3) + C_2(\psi_1 - \psi_2)$.

The method of variation of parameters

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

Assume that we know that $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (\text{hom})$$

Then the general solution of (hom) is

$$c_1 y_1(t) + c_2 y_2(t).$$

Let us try to find a solution of $(*)$ in a form

$$y(t) = c_1(t)\, y_1(t) + c_2(t)\, y_2(t),$$ where $c_1(t)$ and $c_2(t)$ are unknown functions.

$$y'(t) = c_1'(t)\, y_1(t) + c_2'(t)\, y_2(t) + c_1(t) y_1'(t) + c_2 y_2'(t)$$

$$y''(t) = c_1''(t) y_1(t) + c_2''(t) y_2(t) + 2\left(c_1'(t) y_1'(t) + c_2'(t) y_2'(t)\right) + c_1(t) y_1''(t) + c_2(t)\, y_2''(t)$$

Suppose that

$$c_1'(t)\,y_1(t) + c_2'(t)\,y_2(t) = 0.$$

Then $y''(t) = \left(c_1(t)y_1'(t) + c_2(t)y_2'(t)\right)' =$

$$= c_1'y_1' + c_2'y_2' + c_1y_1'' + c_2y_2'', \text{ and}$$

$$L(y) = \left(c_1'y_1' + c_2'y_2' + c_1y_1'' + c_2y_2''\right) +$$

$$+ p(t)\left(c_1y_1' + c_2y_2'\right) + q(t)\left(c_1y_1 + c_2y_2\right) =$$

$$= c_1'y_1' + c_2'y_2' = g(t).$$

Therefore we want

$$\begin{array}{l} y_2' \times \\ y_2 \times \end{array} \begin{cases} c_1'(t)\,y_1(t) + c_2'(t)\,y_2(t) = 0 \\ c_1'y_1' + c_2'y_2' = g(t) \end{cases}$$

$$c_1'\left(y_1y_2' - y_1'y_2\right) = -y_2 \cdot g(t)$$

$$\begin{cases} c_1' = \dfrac{-y_2 \cdot g}{W[y_1, y_2]}, \\[2ex] c_2' = \dfrac{y_1 \cdot g}{W[y_1, y_2]} \end{cases}$$

From here we find $C_1(t)$, $C_2(t)$, and a partial solution of $(*)$.

Example

$$y'' - 4y' + 4y = te^{2t}$$

Homogeneous equation:

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0$$

$$r = 2, \quad C_1 e^{2t} + C_2 t e^{2t} \text{ is a general solution.}$$

Let us try to find a solution of a non-homogeneous equation in a form

$$y(t) = C_1(t) e^{2t} + C_2(t) t e^{2t}.$$

$$y'(t) = \overline{C_1' e^{2t} + C_2' t e^{2t}} +$$

$$+ C_1 \cdot 2e^{2t} + C_2 \cdot (e^{2t} + 2te^{2t})$$

Suppose $C_1' e^{2t} + C_2' t e^{2t} = 0$. Then

$$y''(t) = C_1' \cdot 2e^{2t} + C_2' (e^{2t} + 2te^{2t}) +$$

$$+ 4C_1 e^{2t} + C_2 (2e^{2t} + 2e^{2t} + 4te^{2t}),$$

so

$$\begin{cases} C_1' e^{2t} + C_2' t e^{2t} = 0 \\ 2C_1' e^{2t} + C_2' (e^{2t} + 2te^{2t}) = te^{2t} \end{cases}$$

6

$$\begin{cases} C_1' + C_2' \cdot t = 0 \\ 2C_1' + C_2'(1+2t) = t \end{cases}$$

$$\begin{cases} C_2' = t \\ C_1' = -t^2 \end{cases}, \text{ so } \begin{cases} C_1(t) = -\frac{t^3}{3} + \tilde{C}_1 \\ C_2(t) = \frac{t^2}{2} + \tilde{C}_2 \end{cases},$$

so

$$y(t) = \left(\tilde{C}_1 - \frac{t^3}{3}\right)e^{2t} + \left(\tilde{C}_2 + \frac{t^2}{2}\right)te^{2t}$$

is a general solution

(i.e. $-\frac{t^3}{3}e^{2t} + \frac{t^2}{2}te^{2t}$ is a partial solution
$= \frac{t^3}{6}e^{2t}$, i.e.

$$y(t) = C_1 e^{2t} + C_2 t e^{2t} + \frac{t^3}{6}e^{2t}$$

is a general solution).

Example

$$y'' - y = f(t), \quad y(0) = y'(0) = 0$$

$$y'' - y = 0$$

$$C_1 e^{t} + C_2 e^{-t}$$

$$y(t) = C_1(t) e^{t} + C_2(t) e^{-t}$$

$$y'(t) = \underbrace{[C_1' e^{t} + C_2' e^{-t}]}_{=0} + C_1 e^{t} - C_2 \cdot e^{-t}$$

$$y''(t) = [C_1' e^{t} - C_2' e^{-t}] + C_1 e^{t} + C_2 e^{-t},$$

$$\begin{cases} C_1' e^{t} - C_2' e^{-t} = f(t) \\ C_1' e^{t} + C_2' e^{-t} = 0 \end{cases}$$

$$\begin{cases} C_1'(t) = \frac{1}{2} f(t) \cdot e^{-t} \\ C_2'(t) = -\frac{1}{2} e^{t} f(t) \end{cases}$$

$$\begin{cases} C_1(t) = \frac{1}{2} \int f(t) e^{-t} dt + \tilde{C}_1 \\ C_2(t) = -\frac{1}{2} \int f(t) e^{t} dt + \tilde{C}_2 \end{cases}$$

$$y(t) = e^{t}\left(\tilde{C}_1 + \frac{1}{2}\int_0^t f(t) e^{-t} dt\right) + e^{-t}\left(\tilde{C}_2 - \frac{1}{2}\int_0^t f(t) e^{t} dt\right)$$

$$y(0) = 0 = \tilde{C}_1 + \tilde{C}_2$$

$$y'(0) = e^{t}\left(\tilde{C}_1 + \frac{1}{2}\int_0^t f(t) e^{-t} dt\right) - e^{-t}\left(\tilde{C}_2 - \frac{1}{2}\int_0^t f(t) e^{t} dt\right) +$$

$$+ \underbrace{e^{t} \cdot \frac{1}{2} f(t) e^{-t} - \frac{1}{2} e^{-t} f(t) e^{t}}_{0} = \tilde{C}_1 - \tilde{C}_2 = 0$$

Therefore we need a folution

$$y(t) = \frac{1}{2} e^{t} \int_0^t f(\tau) e^{-\tau} d\tau - \frac{1}{2} e^{-t} \int_0^t f(\tau) e^{\tau} d\tau =$$

$$= \frac{1}{2} \int_0^t f(\tau) \left(e^{t-\tau} - e^{\tau - t} \right) d\tau =$$

$$\boxed{y(t) = \int_0^t f(\tau)\, \mathrm{sh}(t-\tau)\, d\tau}$$

Indeed,

$$y' = f(t) \cdot 0 + \int_0^t f(\tau)\, \mathrm{ch}(t-\tau)\, d\tau$$

$$y'' = f(t) + \underbrace{\int_0^t f(\tau)\, \mathrm{sh}(t-\tau)\, d\tau}_{y}$$