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Introduction to differential equations.

Second-order linear differential equations with constant coefficients.

(*) $a y'' + b y' + c y = 0$

Characteristic function:

$a r^2 + b r + c = 0$

Case 1

$r_1, r_2 \in \mathbb{R}, \quad r_1 \neq r_2$

General solution $y(t) = C_1 e^{r_1 t} + C_1 e^{r_2 t}$

Case 2

$r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$,

$r_1 = \alpha + i\beta$

$r_2 = \alpha - i\beta$

General solution $y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$

Case 3

$r_1 = r_2 \in \mathbb{R} \quad (b^2 - 4ac = 0)$

General solution: $y(t) = C_1 e^{rt} + C_2 e^{rt} \cdot t$

Reduction of order

Consider an equation

$$y'' + p(t)y' + q(t)y = 0, \quad p(t), q(t) \text{ are continuous func.}$$

Assume that $y_1(t) \not\equiv 0$ is a solution.

Let us try to find another solution in the form $y(t) = \varphi(t)\cdot y_1(t)$.

We have:

$$y' = \varphi' \cdot y_1(t) + \varphi(t)\cdot y_1'(t)$$

$$y'' = \varphi'' \cdot y_1(t) + 2\varphi'(t)y_1'(t) + \varphi(t)\cdot y_1''(t),$$

$$y'' + p(t)y' + q(t)y = (\varphi'' y_1(t) + 2\varphi' y_1' + \varphi y_1'') + p(t)(\varphi' y_1 + \varphi y_1') + q(t)y_1\varphi =$$

$$= \varphi \cdot \underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{=0} + \varphi'' \cdot y_1 + 2\varphi' y_1' + p(t)\varphi' y_1 =$$

$$= y_1(t)\cdot\varphi'' + (2y_1' + p(t)\cdot y_1)\varphi' = 0$$

Denote $u(t) = \varphi'$, then

$$y_1(t)u' + (2y_1' + p(t)y_1)u = 0$$

– separable equation.

Find $u(t)$ and $\varphi(t) = \int u(t)dt$,

set $y_2(t) = \varphi(t)\cdot y_1(t)$.

Example

$$L(y) \equiv (1+t^2)\frac{d^2y}{dt^2} - 2t\,\frac{dy}{dt} + 2y = 0$$

$y_1(t) = t$ is a solution:

$$(1+t^2)\cdot 0 - 2t\cdot 1 + 2t = 0.$$

Let us find a general solution.

Try to find another solution in the form

$$y(t) = t\,\varphi(t),$$

$$y'(t) = \varphi(t) + t\cdot\varphi'$$

$$y''(t) = \varphi'(t) + \varphi'(t) + t\varphi''(t) = 2\varphi' + t\varphi''$$

$$L(t\varphi(t)) = (1+t^2)(2\varphi' + t\varphi'') - \underline{2t(\varphi + t\varphi')} + \underline{2t\varphi} =$$

$$= (1+t^2)(2\varphi' + t\varphi'') - 2t^2\varphi' = 0$$

$\underline{\varphi' = u}$, $\quad (1+t^2)t\cdot u' + 2u = 0$

$$\frac{u'}{u} = -\frac{2}{t(1+t^2)}$$

$$\int\frac{du}{u} = -\int\frac{2\,dt}{t(1+t^2)} + \tilde{C}$$

$$\int\frac{2\,dt}{t(1+t^2)} = \int\frac{2t\,dt}{t^2(1+t^2)} = \int\frac{dt^2}{t^2(1+t^2)} = \int\frac{(1+t^2-t^2)\,dt^2}{t^2(1+t^2)} =$$

$$= \int\frac{d(t^2)}{t^2} - \int\frac{d(t^2)}{1+t^2} = \ln\frac{t^2}{1+t^2} + \tilde{C}$$

(4)

$$\ln|u| = -\ln\frac{t^2}{1+t^2} + \tilde{C}$$

$$\underline{u = C\frac{1+t^2}{t^2}} = C\left(1+\frac{1}{t^2}\right)$$

$$\varphi'(t) = C\left(1+\frac{1}{t^2}\right)$$

$$\varphi(t) = C_1\left(t-\frac{1}{t}\right) + C_2,$$

$$\underline{y(t) = t\,\varphi(t) = C_1(t^2-1) + C_2 t}$$ – <u>general solution.</u>

Let us apply this to the Case 3:

$$ay''+by'+cy=0,$$

$$\underline{b^2-4ac=0}$$

$$ar^2+br+c=0$$

$$r_1 = r_2 = -\frac{b}{2a}$$

$y_1(t) = e^{-\frac{b}{2a}t}$ – solution.

Let us try to find another solution in the form

$$y(t) = \varphi(t)\cdot e^{-\frac{b}{2a}t}$$

$$y' = \varphi' e^{-\frac{b}{2a}t} - \frac{b}{2a}\varphi(t) e^{-\frac{b}{2a}t}$$

$$y'' = \varphi'' e^{-\frac{b}{2a}t} - \frac{b}{a}\varphi'(t) e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}\varphi(t) e^{-\frac{b}{2a}t}$$

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

$$\left(\varphi'' e^{-\frac{b}{2a}t} - \frac{b}{a} \varphi'(t) e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2} \varphi(t) e^{-\frac{b}{2a}t} \right) +$$

$$+ \frac{b}{a} \left(\varphi' e^{-\frac{b}{2a}t} - \frac{b}{2a} \varphi(t) \cdot e^{-\frac{b}{2a}t} \right) + \frac{c}{a} \varphi \cdot e^{-\frac{b}{2a}t} =$$

$$= e^{-\frac{b}{2a}t} \left[\varphi(t) \left(\frac{b^2}{4a^2} - \frac{b^2}{2a^2} + \frac{c}{a} \right) + \right.$$

$$\left. + \varphi'' - \frac{b}{a} \varphi' + \frac{b}{a} \varphi' \right] = e^{-\frac{b}{2a}t} \left[\varphi(t) \frac{\overbrace{b^2 - 2b^2 + 4ac}^{=0}}{4a^2} + \right.$$

$$\left. + \varphi'' \right] =$$

$$= e^{-\frac{b}{2a}t} \cdot \varphi''(t) = 0$$

$$\varphi''(t) = 0$$

$$\varphi(t) = C_1 t + C_2$$

$$y(t) = e^{-\frac{b}{2a}t} (C_1 t + C_2), \text{ – general solution,}$$

$$e^{-\frac{b}{2a}t}, \quad t \cdot e^{-\frac{b}{2a}t} \text{ – linearly independent solutions,}$$

$$W\left[e^{-\frac{b}{2a}t}, t\, e^{-\frac{b}{2a}t} \right] = e^{-\frac{b}{a}t} \neq 0.$$

Example

1) $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$\underline{r = 2}$, therefore e^{2t} is a solution, and since $b^2 - 4ac = 0$, $t \cdot e^{2t}$ is also a solution,

i.e. $(C_1 + tC_2)e^{2t}$ is a general solution

$$(te^{2t})'' - 4(te^{2t})' + 4 \cdot te^{2t} =$$

$$= (e^{2t} + 2te^{2t})' - 4(e^{2t} + 2e^{2t} \cdot t) + 4te^{2t} =$$

$$= 2e^{2t} + 2e^{2t} + \cancel{4}te^{2t} - 4e^{2t} - 8e^{2t} \cdot t + 4te^{2t} \equiv 0$$

2) Find a solution of $y'' - 10y' + 25y = 0$ such that

$y(0) = 0$

$y'(0) = 3$

$r^2 - 10r + 25 = 0$

$\underline{r_1 = r_2 = 5}$,

$y(t) = (C_1 + tC_2)e^{5t}$ is a general solution

$y(0) = C_1 = 0$

$y'(t) = 5C_1e^{5t} + C_2e^{5t} + 5tC_2e^{5t}$,

$y'(0) = 5C_1 + C_2 = 3$,

so $\underline{C_1 = 0,\ C_2 = 3}$

Answer: $\underline{y(t) = 3te^{5t}}$

Non-homogeneous equations

$$a y'' + b y' + c y = g(t)$$

or

(*) $$y'' + p(t) y' + q(t) y = g(t) \quad (\Leftrightarrow L(y) = g(t))$$

Theorem

Let $p(t)$, $q(t)$ be continuous on (α, β). Then if $y_1(t)$, $y_2(t)$ are linearly independent solutions of a homogeneous equation

(#) $$y'' + p(t) y' + q(t) y = 0,$$

and $\psi(t)$ is a solution of (*) then a general solution of (*) has the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \psi(t)$$

Proof

Assume that $\psi_1(t)$ and $\psi_2(t)$ are solutions of (*), then

$$L(\psi_1 - \psi_2) = L(\psi_1) - L(\psi_2) = g(t) - g(t) = 0,$$

so $\psi_1 - \psi_2$ is a solution of (#).

Any solution of (#) has the form

$C_1 y_1(t) + C_2 y_2(t)$, so any solution of (*) has the form $C_1 y_1(t) + C_2 y_2(t) + \psi(t)$.

$\square$

<u>Example</u>

$$y'' - 3y' + 2y = 2t - 3$$

Notice that $\psi(t) = t$ is a partial solution.

A general solution of

$y'' - 3y' + 2y = 0$ has the form

$$c_1 e^{t} + c_2 e^{2t},$$

so a general solution of the initial equation is

$$\underline{c_1 e^{t} + c_2 e^{2t} + t}.$$