Introduction to differential equations.

Second order linear differential equations with constant coefficients.

$L : y \mapsto ay'' + by' + cy, \quad Ly = 0$

We need to find two linearly independent solutions $y_1(t), y_2(t)$. Then general solution is of the form $C_1 y_1(t) + C_2 y_2(t)$.

Try $y(t) = e^{rt}$,

$$L(e^{rt}) = ar^2 e^{rt} + bre^{rt} + ce^{rt} =$$

$$= (ar^2 + br + c)e^{rt} = 0$$

$\underline{ar^2 + br + c = 0}$ characteristic equation

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If r_1, r_2 are real and distinct then

$y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$

are linearly independent solutions,

and $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is a general solution.

If $b^2-4ac<0$ then r_1, r_2 are complex numbers. ②

$z \in \mathbb{C} \iff z = u+iv, \quad i^2=-1$

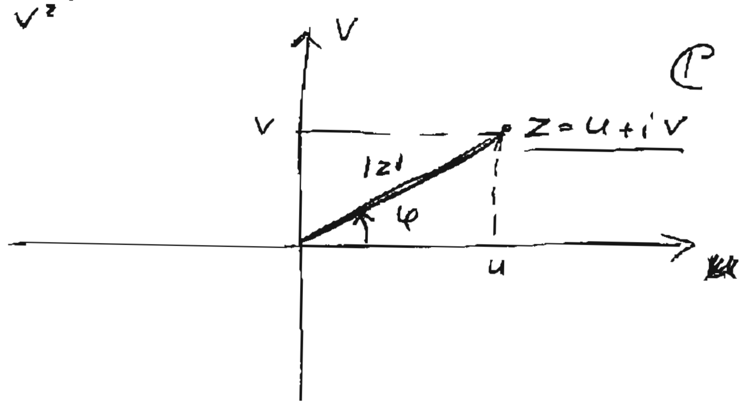
$z_1 = u_1 + i v_1, \quad z_2 = u_2 + i v_2,$

$z_1 + z_2 = (u_1+u_2) + i(v_1+v_2),$

$z_1 \cdot z_2 = (u_1+iv_1)\cdot(u_2+iv_2) = (u_1 u_2 - v_1 v_2) + i(u_1 v_2 + v_1 u_2)$

$z = u+iv \Rightarrow \bar{z} = u-iv,$

$|z| = \sqrt{z\bar{z}} = \sqrt{u^2+v^2}$



$u = |z|\cos\varphi$
$v = |z|\sin\varphi$

$$\underline{z = |z|e^{i\varphi}}$$

Explanation:

$$e^{i\varphi} = 1 + (i\varphi) + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \dots =$$

$$= \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots\right) + i\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots\right) =$$

$$= \cos\varphi + i\sin\varphi$$

$$u + iv = |z|\cos\varphi + i|z|\sin\varphi = |z|(\cos\varphi + i\sin\varphi) =$$

$$= \underline{|z|e^{i\varphi}}$$

Polar representation of a complex number.

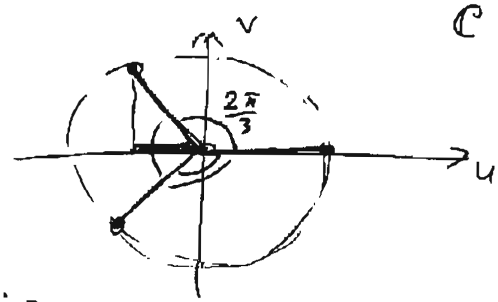
$$z_1 = |z_1| e^{i\varphi_1}, \quad z_2 = |z_2| \cdot e^{i\varphi_2},$$

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot e^{i(\varphi_1 + \varphi_2)}$$

Example

$$x^3 = 8$$

$$x_1 = 2$$



$$z = |z| e^{i\varphi}, \quad z^3 = |z|^3 e^{i \cdot 3\varphi} = 8$$

$$\Rightarrow |z| = 2, \quad e^{i \cdot 3\varphi} = 1$$

$$x_2 = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$x_3 = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

Notation

$Re(z) = u$

$Im(z) = v,$

if $z = u + iv$

Complex-valued functions:

$$y(t) : (\alpha, \beta) \to \mathbb{C}$$

$$y(t) = u(t) + i\,v(t), \quad u : (\alpha, \beta) \to \mathbb{R}^1$$

$$v : (\alpha, \beta) \to \mathbb{R}^1.$$

$$y'(t) = u'(t) + i\,v'(t)$$

Lemma

If $y(t) = u(t) + i\,v(t)$ is a complex-valued solution of $ay'' + by' + cy = 0$, $a, b, c \in \mathbb{R}$. Then $u(t)$ and $v(t)$ are solutions too.

Therefore, if

$$e^{rt} \text{ is a solution of } (*), \quad r = \alpha + i\beta,$$

then

$$e^{(\alpha+i\beta)t} = e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} =$$

$$= e^{\alpha t}(\cos\beta t + i \sin\beta t), \text{ and}$$

$$\operatorname{Re} e^{rt} = e^{\alpha t} \cdot \cos\beta t,$$

$$\operatorname{Im} e^{rt} = e^{\alpha t} \cdot \sin\beta t.$$

Remarks

1)

$$W[e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t] =$$

$$= \det \begin{pmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ \alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t & \alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t \end{pmatrix} =$$

$$= e^{2\alpha t}\Big(\cancel{\alpha\cos\beta t \sin\beta t} + \beta\cos^2\beta t \} - \cancel{\alpha\cos\beta t \sin\beta t} + \beta \sin^2\beta t\Big) =$$

$$= \beta \cdot e^{2\alpha t} \neq 0 \text{ if } \beta \neq 0,$$

so these solutions are linearly independent! (if $\beta \neq 0$).

2) If r is a complex root of $ar^2+br+c=0$, (5)

then $\bar{r} = \alpha - i\beta$ is a root too.

But $e^{\bar{r}t} = e^{\alpha t}(\cos\beta t - i \sin\beta t)$, and

we get the same pair of solutions.

Example

1) $y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$e^{it} = (\cos t + i \sin t)$

$Re\ e^{it} = \cos t = y_1$

$Im\ e^{it} = \sin t = y_2$,

so $C_1 \cos t + C_2 \sin t$ is a general solution.

2) $y'' - 2y' + 5y = 0$

$r^2 - 2r + 5 = 0$ - characteristic equation.

$r_{1,2} = \{1+2i, 1-2i\}$

$r_1 = 1+2i, \quad e^{r_1 t} = e^{(1+2i)t} = e^t \cdot e^{2it} =$

$= e^t(\cos 2t + i \sin 2t)$,

so $C_1 e^t \cos 2t + C_2 e^t \sin 2t$

is a general solution.

Equal roots

If $b^2 - 4ac = 0$ then

$$e^{-\frac{b}{2a}t} \text{ is a solution}$$

In order to find a second solution try $y(t) = \varphi(t) \cdot e^{-\frac{b}{2a}t}$.

After a substitution we get

$$e^{-\frac{b}{2a}t} \cdot \varphi''(t) = 0,$$

$$\varphi(t) = C_1 + C_2 t,$$

so $y(t) = e^{-\frac{b}{2a}t}(C_1 + C_2 t)$ is a general solution

Remark

$e^{\alpha t}$ and $e^{\alpha t} \cdot t$ are linearly independent,

$$\det \begin{pmatrix} e^{\alpha t} & e^{\alpha t} \cdot t \\ \alpha e^{\alpha t} & \alpha e^{\alpha t} \cdot t + e^{\alpha t} \end{pmatrix} =$$

$$= e^{2\alpha t}(\alpha t + 1 - \alpha t) = e^{2\alpha t} \neq 0.$$

Reduction of order in general case

Assume that $y_1(t)$ is a solution of

$$y'' + p(t)y' + q(t)y = 0.$$

try $y(t) = \varphi(t)\cdot y_1(t)$.

$$y' = \varphi' y_1 + \varphi\cdot y_1'$$

$$y'' = \varphi''\cdot y_1 + \varphi' y_1' + \varphi' y_1' + \varphi y_1'' = \varphi''\cdot y_1 + 2\varphi' y_1' + \varphi y_1''$$

$$\varphi''\cdot y_1 + 2\varphi' y_1' + \varphi\cdot y_1'' + p(t)\,(\varphi' y_1 + \varphi\cdot y_1') + q(t)\varphi\cdot y_1 = 0$$

$$\varphi\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{=0} + \varphi''\cdot y_1 + 2\varphi' y_1' + p(t)\varphi' y_1 = 0$$

$$\varphi''\cdot y_1 + 2\varphi' y_1' + p(t)\varphi'\cdot y_1 = 0$$

Denote $\varphi'(t) = u(t)$

$$u'\cdot y_1(t) + [2y_1' + p(t)y_1]\,u = 0$$

$$\frac{u'}{u} = -\frac{[2y_1' + p(t)y_1]}{y_1}, \text{ so}$$

$$u(t) = C\exp\left[-\int\left(2\frac{y_1'}{y_1} + p(t)\right)dt\right] =$$

$$= \frac{c\exp\left[-\int p(t)dt\right]}{y_1^2(t)}$$

Take $C=1$, $u(t)=\frac{1}{y_1^2(t)}\exp\left[-\int p(t)dt\right]$,

and $\varphi(t)=\int u(t)dt$.

$$y_2(t)=\varphi(t)\,y_1(t)=y_1(t)\int u(t)dt.$$

Example

1) $y''-2y'+y=0$

$r^2-2r+1=0$

$r_{1,2}=1$, $r_1=r_2$.

e^t is a solution, $t\cdot e^t$ is also a solution,

so $(C_1+C_2t)e^t$ is a general solution

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2) Solve $(1+t^2)y''-2ty'+2y=0$.

Hint: $y_1=t$ is a solution.

Let us try to find another solution in the form $y(t)=\varphi(t)\cdot t$

$y'(t)=\varphi'\cdot t+\varphi$

$y''(t)=\varphi''\cdot t+2\varphi'$

$(1+t^2)(\varphi''\cdot t+2\varphi')-2t(\varphi'\cdot t+\varphi)+(2\cdot\varphi\cdot t)=0$

$\varphi''\cdot t(1+t^2)+2(1+t^2)\varphi'-2t^2\cdot\varphi'-\cancel{2t\varphi}+\cancel{2t\varphi}=0$

$\varphi''\cdot t(t^2+1)+2\varphi'=0$

$u(t) = \varphi'(t)$, so

$$u'(t) \cdot t(t^2+1) + 2u = 0$$

$$\frac{u'}{u} = -\frac{2}{t(t^2+1)} = -\frac{2(t^2+1-t^2)}{t(t^2+1)} =$$

$$= -\frac{2}{t} + \frac{2t}{t^2+1}$$

$$\ln|u| = -2\ln|t| + \int \frac{2t\,dt}{t^2+1} =$$

$$= -2\ln|t| + \int \frac{dt^2}{t^2+1} =$$

$$= \cancel{2}-\ln|t|^2 + \ln(t^2+1) =$$

$$= \ln\frac{t^2+1}{t^2}$$

$$u = C \cdot \frac{t^2+1}{t^2} = C\left(1+\frac{1}{t^2}\right)$$

$$\varphi'(t) = C\left(1+\frac{1}{t^2}\right)$$

$$\varphi(t) = C_1\left(t - \frac{1}{t}\right) + C_2,$$

$$y(t) = t\left(C_1\left(t - \frac{1}{t}\right) + C_2\right) =$$

$$= \underline{C_2 t + C_1(t^2-1)}$$ - general solution