Introduction to differential equations.

Linear differential equations of second order.

$y'' + p(t)y' + q(t)y = g(t)$ – non-homogeneous

(*) $\boxed{y'' + p(t)y' + q(t)y = 0}$ – homogeneous

$L : y \mapsto y'' + p(t)y' + q(t)y$ – linear operator on $C^2(\alpha,\beta)$,

$\boxed{Ly = 0}$

Space of solutions ($\equiv \operatorname{Ker} L$) is a linear space

Thm
Let y_1, y_2 be two solutions of (*) on (α,β), and $p(t), q(t)$ be continuous on (α,β). Then

if $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_2y_1' \equiv W[y_1,y_2](t) \neq 0 \quad \forall t \in (\alpha,\beta)$

then $\underline{C_1y_1(t) + C_2y_2(t)}$ is a general solution of (*).

Thm (on Wronskian)

If $p(t), q(t)$ are continuous on (α,β), and y_1, y_2 are solutions of (*) on (α,β), then

either $W[y_1,y_2] \equiv 0$,

or $W[y_1,y_2](t) \neq 0 \quad \forall t \in (\alpha,\beta)$.

Def Two functions $y_1(t)$ and $y_2(t)$, $t \in (\alpha, \beta)$, are linearly dependent if for some C_1, C_2, $|C_1| + |C_2| \neq 0$,

$$C_1 y_1(t) + C_2 y_2(t) = 0 \quad \forall t \in (\alpha, \beta)$$

($\equiv$ one function is a constant multiple of another).

Def n functions $y_1(t), \ldots, y_n(t)$, $t \in (\alpha, \beta)$, are linearly dependent if $\exists\, C_1, \ldots, C_n$, $|C_1| + \ldots + |C_n| \neq 0$, s.t.

$$C_1 y_1(t) + \ldots + C_n y_n(t) \equiv 0 \text{ on } (\alpha, \beta).$$

Proposition

Two solutions of $(*)$ are linearly dependent $\iff W[y_1, y_2](t) \equiv 0$.

Proof

$(\Rightarrow)$ Easy: $y_1(t) = C y_2(t)$,

$$\det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det \begin{pmatrix} C y_2 & y_2 \\ C y_2' & y_2' \end{pmatrix} = 0$$

$(\Leftarrow)$ Assume

$$y_1(t) y_2'(t) - y_1'(t) y_2(t) = 0 \quad \forall t \in (\alpha, \beta)$$

If $y_1(t) y_2(t) \neq 0 \ \forall t \in (\alpha, \beta)$ then

$$\frac{y_1'(t)}{y_1(t)} = \frac{y_2'(t)}{y_2(t)} \quad \forall t \in (\alpha, \beta), \text{ and}$$

$$\int \frac{y_1'}{y_1}\,dt = \int \frac{y_2'}{y_2}\,dt$$

$$\ln|y_1| = \ln|y_2| + \tilde{C}$$

$|y_1| = e^{\tilde{C}}\cdot|y_2|$, and since y_1 and y_2 do not have zeros on (α,β),

$y_1 = C\,y_2$ for some $C \in \mathbb{R}$.

If $\exists\, t_0 \in (\alpha,\beta)$ s.t. $y_1(t_0)\cdot y_2(t_0) = 0$ then

WLOG assume that $y_1(t_0) = 0$.

Then $y_2(t_0)\cdot y_1'(t_0) \overset{=0}{\cancel{\neq 0}}$, so ~~$y_2(t_0) \neq 0$.~~

$y_2(t_0) = 0$ or $y_1'(t_0) = 0$.

If $y_1'(t_0) = 0$ (and $y_1(t_0) = 0$) then $y_1(t) \equiv 0$, and $y_1(t) = 0\cdot y_2$.

If $y_1'(t_0) \neq 0$ then $y_2(t_0) = 0$.

Consider $\varphi(t) = \dfrac{y_2'(t_0)}{y_1'(t_0)}\, y_1(t)$.

$*$

Then
$$\left.\begin{aligned}\varphi(t_0) &= \frac{y_2'(t_0)}{y_1'(t_0)}\, y_1(t_0) = 0 = y_2(t_0)\\ \varphi'(t_0) &= \frac{y_2'(t_0)}{y_1'(t_0)}\, y_1'(t_0) = y_2'(t_0)\end{aligned}\right\} \Rightarrow \varphi(t) \equiv y_2(t),$$

so $y_2(t) = \left[\dfrac{y_2'(t_0)}{y_1'(t_0)}\right]\cdot y_1(t)$. $\square$.

Therefore if y_1 and y_2 are solutions of $(*)$ then

$C_1 y_1 + C_2 y_2$ is a general solution $\iff W[y_1, y_2] \neq 0 \iff y_1$ and y_2 are linearly independent

Example

If $a \neq b$ then e^{at} and e^{bt} are linearly independent functions

Indeed, $c_1 e^{at} + c_2 e^{bt} = e^{at}(c_1 + c_2 e^{(b-a)t}) \not\equiv 0$. $\forall c_1, c_2$.

$$W[e^{at}, e^{bt}] = \det \begin{pmatrix} e^{at} & e^{bt} \\ ae^{at} & be^{bt} \end{pmatrix} = e^{(a+b)t} \cdot (b-a) \neq 0.$$

Example

$y_1(t) = t$, $y_2(t) = e^t$,

$$W[y_1, y_2] = \det \begin{pmatrix} t & e^t \\ 1 & e^t \end{pmatrix} = e^t \cdot t - e^t = e^t(t-1), \text{ so } W[y_1, y_2](1) = 0.$$

Not a contradiction since $\nexists$ $p(t), q(t)$ such that e^t and t are solutions of $(*)$!

Linear homogeneous differential equations of second order with constant coefficients.

Consider an equation

$$(**) \qquad ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}^1$$

Let us try to find a solution of $(**)$ in a form $y(t) = e^{rt}$, $r \in \mathbb{R}^1$.

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$\boxed{ar^2 + br + c = 0}$$

<u>Def</u> An equation $ar^2 + br + c = 0$ is a characteristic equation of the diff. equation $(**)$.

$y(t) = e^{rt}$ is a solution of $(**)$ $\Leftrightarrow$ r is a root of $ar^2 + br + c = 0$.

If r_1, r_2 are to different real roots of $ar^2 + br + c = 0$ then

$y_1(t) = e^{r_1 t}$ is a solution,

$y_2(t) = e^{r_2 t}$ is a solution, and y_1 and y_2 are linearly independent, so

$\underline{C_1 e^{r_1 t} + C_2 e^{r_2 t}}$ is a general solution of $(**)$

Example

1) $$y'' - 3y' + 2 = 0$$

Characteristic equation:

$$r^2 - 3r + 2 = 0$$

$$D = 9 - 8 = 1,$$

$$r_{1,2} = \frac{3 \pm 1}{2} = \{2, 1\}, \text{ so}$$

e^t and e^{2t} are solutions, so

$C_1 e^t + C_2 e^{2t}$ is a general solution.

2) $$y'' - 5y = 0$$

Characteristic equation:

$$r^2 - 5 = 0$$

$$r_{1,2} = \{\sqrt{5}, -\sqrt{5}\}, \text{ so}$$

$e^{\sqrt{5}t}$ & $e^{-\sqrt{5}t}$ are solutions

$C_1 e^{\sqrt{5}t} + C_2 e^{-\sqrt{5}t}$ is a general solution.

3) Find a solution of

$$\cancel{x^2} \; y'' - 3y' + 2y = 0$$

such that $\begin{cases} y(0) = 2 \\ y'(0) = 3 \end{cases}$

Solution:

We know that a general solution has the form $y(t) = C_1 e^t + C_2 e^{2t}$.

$$\left.\begin{aligned} y(0) &= C_1 + C_2 = 2 \\ y'(0) &= C_1 + 2C_2 = 3 \end{aligned}\right| \Rightarrow C_2 = 1 = C_1, \text{ so}$$

$$y(t) = e^t + e^{2t} \quad \square$$

Example

Euler's equation:

$$y'' + \frac{\alpha}{t} y' + \frac{\beta}{t^2} y = 0, \quad t > 0.$$

Solution:

Try to find $y(t) = t^r$:

$$r(r-1)t^{r-2} + r t^{r-1} \cdot \frac{\alpha}{t} + t^r \frac{\beta}{t^2} = 0$$

$$r(r-1) + r\alpha + \beta = 0$$

$$r^2 + (\alpha - 1)r + \beta = 0$$

Find r_1, r_2 (roots of),

$C_1 t^{r_1} + C_2 t^{r_2}$ - general solution

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0, \quad t > 0$$

$$r^2 - 3r + 2 = 0, \quad r_{1,2} = 1, 2,$$

$$y(t) = C_1 t + C_2 t^2 \quad \text{- general solution}$$