Oct. 10, 2008

Introduction to differential equations

Second-order linear differential equations.

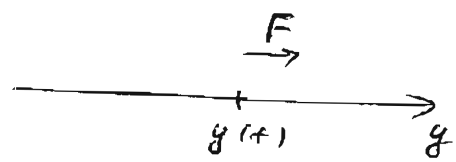
Def

A second order differential equation is an equation of the form

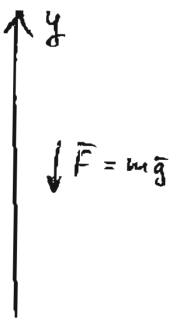
$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Example

1) $$m \cdot \frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)$$



2)



$$\frac{d^2y}{dt^2} = -g, \quad y(t_0) = y_0, \; y'(t_0) = y_0'$$

Initial value problem:

Find a function $y(t)$ such that

$$\begin{cases} \dfrac{d^2y}{dt^2} = f\left(t, y, \dfrac{dy}{dt}\right), \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

Def

Linear differential equation of second order

$$\frac{dy^2}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

- nonhomogeneous

$$\frac{dy^2}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \qquad (*)$$

- homogeneous.

Thm (Existence and uniqueness)

Suppose $p(t)$ and $q(t)$ are continuous on the interval (α, β), $-\infty \le \alpha < \beta \le +\infty$, $t_0 \in (\alpha, \beta)$. Then there exists one, and only one function on (α, β) such that $y(t_0) = y_0$, $y'(t_0) = y_0'$, and $(*)$ holds.

Corollary

If $y(t)$ is a solution of $(*)$, $y(t_0) = y'(t_0) = 0$ then $y(t) \equiv 0$.

Example

$y(t)$ is a solution of $(*)$, for $t \in \mathbb{R}$, $y(\frac{1}{n}) = 0$ $\forall n \in \mathbb{N}$. Find $y(t)$.

Solution: $y \equiv 0$, since $y(0) = 0$ (by cont.), $y'(0) = 0$.

Important remark

(*) can be rewritten as

$L(y) = 0$, where

$L : C^2(\alpha,\beta) \to C^0(\alpha,\beta)$ - a linear operator,

$L(y) = y'' + p(t)y' + q(t)y$,

$C^2(\alpha,\beta) = \{ f : (\alpha,\beta) \to \mathbb{R} \mid \exists f''$, and f'' is continuous on $(\alpha,\beta)\}$

$C^0(\alpha,\beta) = \{ f : (\alpha,\beta) \to \mathbb{R},\ f$ is continuous$\}$.

Since $L(y_1+y_2) = (y_1+y_2)'' + p(t)(y_1+y_2)' + q(t)(y_1+y_2)$

$= y_1'' + p(t)y_1' + q(t)y_1 +$

$+ y_2'' + p(t)y_2' + q(t)y_2 =$

$= L(y_1) + L(y_2)$,

and $L(cy) = (cy)'' + p(t)(cy)' + q(t)(cy) =$

$= C\,L(y)$,

L is a linear operator.

Example

Consider an operator $L(y) = y'' + t^2 y'$,

Then $L(e^t) = e^t + t^2 e^t$

$L(e^{-t}) = e^{-t} - t^2 e^{-t}$

$L(t) = t^2$

$L(t^2) = 2 + 2t^3$, $\quad L(1) = 0$

y is solution of $(*)$ $\iff$ $L(y) = 0$,

Remark

Solutions of $(*)$ = Ker L.

(Def: $\operatorname{Ker} L = \{ y \in C^2(\alpha,\beta) \mid Ly = 0 \}$)

Lemma

The space of solutions of $(*)$ is a linear space.

Proof

$L(y_1) = 0$, $L(y_2) = 0 \Rightarrow L(y_1 + y_2) = L(y_1) + L(y_2) = 0$,

so $y_1 + y_2$ is a solution too.

$L(cy_1) = cL(y_1) = 0 \Rightarrow cy_1$ is a solution too $\square$.

Example

$$y'' + y = 0$$

$y_1(t) = \cos t$ is a solution

$y_2(t) = \sin t$ is a solution

$y(t) = C_1 \cos t + C_2 \sin t$ is a solution $\forall C_1, C_2 \in \mathbb{R}$!

Proposition

Any solution of $y'' + y = 0$ has the form

$C_1 \cos t + C_2 \sin t$.

Proof

Assume that $y(t)$ is a solution of the equation

$y''+y=0$

Let $y(0)=y_0$, $y'(0)=y_0'$, and take

$$\tilde{y}(t)=y_0\cos t+y_0'\sin t$$

Then $\tilde{y}(0)=y_0$

$$\tilde{y}'(0)=-y_0\sin 0+y_0'\cos 0=y_0'$$

But $\exists$ only one solution such that

at 0 it is y_0 and its derivative is y_0', so

$y(t)\equiv\tilde{y}(t)$ $\square$.

Theorem

Let $y_1(t)$ and $y_2(t)$ be two solutions of $(*)$ on (α,β), ~~$t_0\in(\alpha,\beta)$~~ and

$$\det\begin{pmatrix} y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{pmatrix}\equiv y_1(t)y_2'(t)-y_1'(t)y_2(t)$$

does not vanish $\forall t\in(\alpha,\beta)$.

Then

$$y(t)=C_1y_1(t)+C_2y_2(t)$$

is a general solution of $(*)$.

Def An expression $\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$ is called the Wronskian of y_1 and y_2.

Notation: $W(t) = W[y_1, y_2](t)$.

Proof

Let $y(t)$ be a solution of $(*)$.

Take any $t_0 \in (\alpha, \beta)$, and let y_0, y_0' be the values of $y(t_0)$ and $y'(t_0)$.

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 & \Big| \times y_2'(t_0) \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' & \Big| \times y_2(t_0) \end{cases}$$

$$c_1 W[y_1, y_2](t_0) = y_0 \cdot y_2'(t_0) - y_0' \cdot y_2(t_0),$$

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' \cdot y_2(t_0)}{W[y_1, y_2](t_0)}$$

Similarly,

$$c_2 = \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{W[y_1, y_2](t_0)}.$$

Set $\varphi(t) = c_1 y_1(t) + c_2 y_2(t)$.

Then $\varphi(t_0) = y_0$

$\varphi'(t_0) = y_0'$, so $\varphi(t) \equiv y(t)$ $\square$

Theorem

Let $p(t)$, $q(t)$ be continuous on (α, β), and y_1, y_2 be two solutions of $(*)$.

Then either $W[y_1, y_2] \equiv 0$ or

$$W[y_1, y_2](t) \neq 0 \quad \forall t \in (\alpha, \beta).$$

Proof

$W[y_1, y_2](t) = y_1(t) y_2'(t) - y_2(t) y_1'(t)$, so

$$\frac{d}{dt} W(t) = \cancel{y_1'(t) y_2'(t)} + y_1(t) y_2''(t) - $$

$$- \cancel{y_2'(t) y_1'(t)} - y_2(t) y_1''(t) =$$

$$= y_1(t)\left(-p(t) y_2'(t) - q(t) y_2(t)\right) -$$

$$- y_2(t)\left(-p(t) y_1'(t) - q(t) y_1(t)\right) =$$

$$= -p(t) W(t)$$

Take any $t_0 \in (\alpha, \beta)$, then

$$W[y_1, y_2](t) = W[y_1, y_2](t_0) \cdot \underbrace{e^{-\int_{t_0}^{t} p(s)\,ds}}_{\neq 0},$$

therefore either $W(t) \equiv 0$ or

$$W(t) \neq 0 \quad \forall t \in (\alpha, \beta). \qquad \square$$

Remark

If $y_1 \equiv 0$ then $W[y_1, y_2] \equiv 0$.

If $y_1 = c y_2$ then $W[y_1, y_2] \equiv 0$.

Def

Functions $y_1(t), y_2(t)$ are linearly dependent if $y_1 = C y_2$ (or $y_2 = C y_1$).

Theorem

If y_1, y_2 are two solutions, and at some point $t_0 \in (\alpha, \beta)$ $W(t_0) = 0$ then y_1 and y_2 are linearly dependent.

Proof

$$W[y_1, y_2](t_0) = 0 \implies W(t) \equiv 0, \text{ so}$$

$$y_1(t) y_2'(t) \equiv y_1'(t) y_2(t).$$

If $y_1(t) y_2(t) \neq 0$ for $\alpha < t < \beta$ then

$$\frac{y_2'(t)}{y_2(t)} = \frac{y_1'(t)}{y_1(t)}$$

$$(\ln |y_2(t)|)' = (\ln |y_1(t)|)'$$

$$\ln |y_2(t)| = \ln |y_1(t)| + \tilde{c}$$

$$|y_2(t)| = e^{\tilde{c}} \cdot |y_1(t)|$$

$$y_2(t) = c\, y_1(t).$$

If $\exists\, t^*$ s.t. $y_1(t^*)=0$ then

$$y_1'(t^*)\, y_2(t^*)=0$$

If $y_1'(t^*)=0$ then $y_1 \equiv 0$.

If $y_1'(t^*)\neq 0$ then $y_2(t^*)=0$,

and $y_2'(t^*) = \left[\dfrac{y_2'(t^*)}{y_1'(t^*)}\right]\cdot y_1'(t^*)$.

Consider $\varphi(t) = \left[\dfrac{y_2'(t^*)}{y_1'(t^*)}\right]\cdot y_1(t)$ - solution of $(*)$.

We have $\varphi(t^*) = 0 = y_2(t^*)$

$$\varphi'(t^*) = y_2'(t^*),$$

then $\varphi(t) \equiv y_2(t)$, so

y_1 and y_2 are linearly dependent $\square$

<u>Example</u>

1) ~~Prove that~~ e^{at}, e^{bt} are linearly independent if $a\neq b$.

2) t, t^3 — " —

3) $\cos t, \sin t$ — " —

4) $t^2, t\cdot|t|$ are linearly independent on $\mathbb{R}$, but $W(t)\equiv 0$. These functions are not solutions of $\underbrace{\text{(}*\text{)}}_{\text{(the same)}}$ for any $p(t), q(t)$!