Lecture 3

Oct. 1, 2008

Introduction to differential equations

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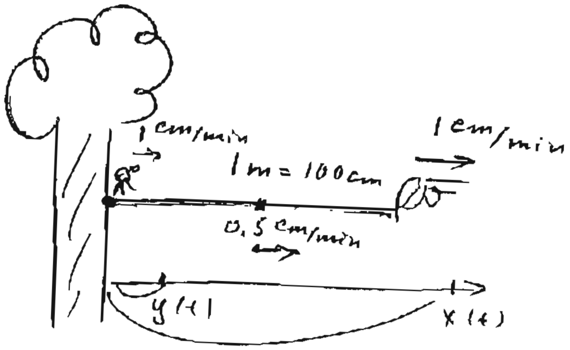
Def

$$y' + a(t)y = b(t)$$

linear differential equation of first order.



$x(t)$ - position of the right side of the rubber band

$y(t)$ - position of the bug.

$$x(t) = 100 + t$$

$$\boxed{y'(t) = 1 + \frac{y}{x} = 1 + \frac{y}{100+t}}$$

Homogeneous:

$$y' = \frac{y}{100+t}$$

$$\frac{y'}{y} = \frac{1}{100+t}$$

$$\ln|y| = \ln|100+t| + \tilde{c}$$

$$y = C(100+t)$$

Non-homogeneous:

$$y' = \frac{y}{100+t} + 1,$$

$$y = \varphi(t)\cdot(100+t)$$

$$\varphi'(t)(100+t) + \cancel{\varphi(t)} = \cancel{\varphi(t)} + 1$$

$$\varphi'(t) = \frac{1}{100+t} \Rightarrow \underline{\varphi(t) = \ln|100+t| + C}$$

$$y(t) = (100+t)(C + \ln(100+t)).$$

$y(0)=0$

$$0 = 100(C + \ln 100)$$

$$C = -\ln 100$$

$$\boxed{y(t) = (100+t)\ln\left(1+\frac{t}{100}\right)}$$

$$y(t) = x(t) \ ?$$

$$x(t) = 100+t = (100+t)\ln\left(1+\frac{t}{100}\right)$$

$$\ln\left(1+\frac{t}{100}\right) = 1$$

$$1 + \frac{t}{100} = e = 2{,}7\ldots$$

$$t = 100(e-1) > 170 \text{ (min)}$$

Separable equations

$$\frac{dy}{dt} = \frac{g(t)}{f(y)} \quad , g, f \text{ are continuous functions.}$$

Remark Not every function $f(t, y)$ can be represented as $\frac{g(t)}{f(y)}$!

$$f(y)\frac{dy}{dt} = g(t)$$

Assume that $F(y)$ is such that $F'(y) = f(y)$.

Then $\frac{d}{dt}F(y(t)) = f(y)\,y'$ $\qquad$ $F(y) = \int f(y)\,dy + C$

Therefore

$$\boxed{\int f(y)\,dy = \int g(t)\,dt + C}$$

general solution.

If we want to solve an initial-value problem $y(t_0) = y_0$, then

$$\boxed{\int_{y_0}^{y} f(y)\,dy = \int_{t_0}^{t} g(t)\,dt}$$

Example

$$\frac{dy}{dt} = \frac{t}{y}$$

$$y\frac{dy}{dt} = t$$

$$\int y\,dy = \int t\,dt + C$$

$\boxed{y^2 = t^2 + C}$ general solution

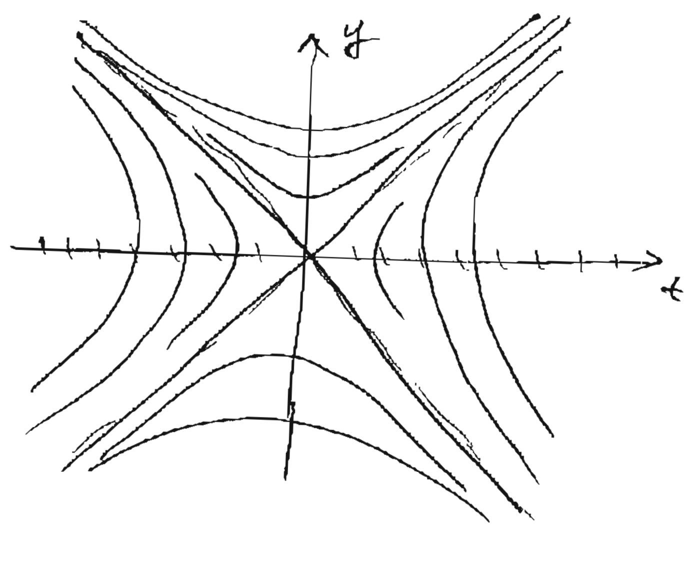
Assume $y(0) = 1$

$$y^2(0) = 1 = 0^2 + C\,, \quad C = 1$$

$$y^2 = t^2 + 1$$

$$y = \pm\sqrt{t^2+1}$$

$$y = \sqrt{t^2+1}$$



$$\frac{dy}{dt} = 10^{y+t} \qquad\qquad 10^{-y} = e^{-y\ln 10}$$

$$\int 10^{-y}\,dy = \int 10^{t}\,dt + \tilde{C}$$

$$\int 10^{-y}\,dy = \int e^{-y\ln 10}\,dy = -\frac{1}{\ln 10}e^{-y\ln 10} = -\frac{10^{-y}}{\ln 10}$$

$$\int 10^{t}\,dt = \frac{10^{t}}{\ln 10}$$

$$-\frac{10^{-y}}{\ln 10} = \frac{10^{t}}{\ln 10} + \tilde{C}$$

$$\boxed{10^{-y} + 10^{t} = C} \qquad \boxed{y = -\frac{\ln(C - 10^{t})}{\ln 10}}$$

Remark

Interval of validity for an initial value problem

$y(t_0) = y_0$
$y' = f(t, y)$

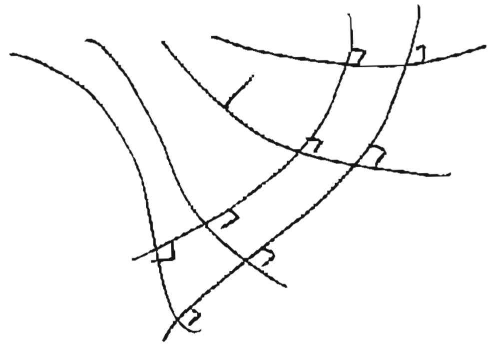
is the largest possible interval on which the solution is valid and contains t_0.

Example

Find the orthogonal trajectories of the family of curves

$$F(x, y, c) = 0 \,, \quad c \in \mathbb{R}^1.$$

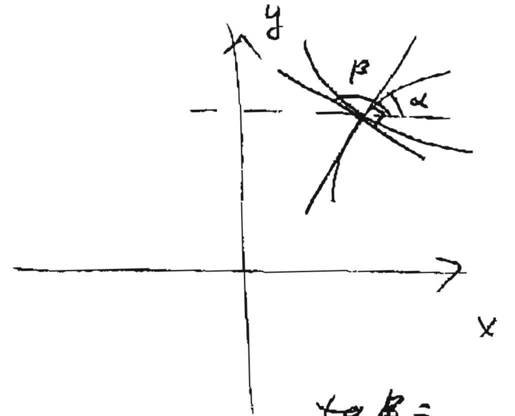
Solution:



$$\frac{d}{dx} F(x, y, c) = 0$$

$$\left(\frac{\partial}{\partial x} F\right) + \left(\frac{\partial}{\partial y} F\right) \cdot \frac{dy}{dx} = 0 \,,$$

$$y' = -\frac{F_x}{F_y}$$



$$F(x, y, c) = 0 \Rightarrow c = c(x, y)$$

Therefore

$y' = \dfrac{F_y}{F_x}$ is an equation that describes the curves orthogonal to $F(x,y,c)=0$.

$$\operatorname{tg}\beta = \operatorname{tg}\left(\alpha + \frac{\pi}{2}\right) = -\frac{1}{\operatorname{tg}\alpha}$$

For example, find the orthogonal trajectories of the family of parabolas

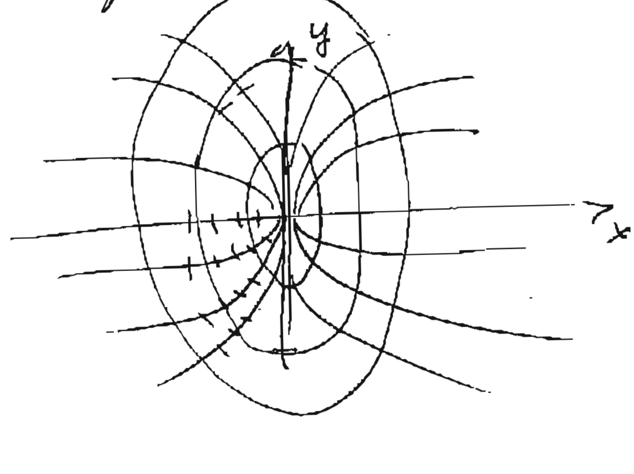
$$x = c\, y^2$$

$$1 = 2 c y y' \,, \quad c = \frac{x}{y^2} \,,$$

$$1 = 2\, \frac{x}{y^2} \cdot y y'$$

$$y' = \frac{y}{2x}$$

$$y = y(x)$$



The equation that describes the orthogonal family is

$$y' = -\frac{2x}{y}$$

$$\int y\,dy = \int -2x\,dx + \tilde{C}$$

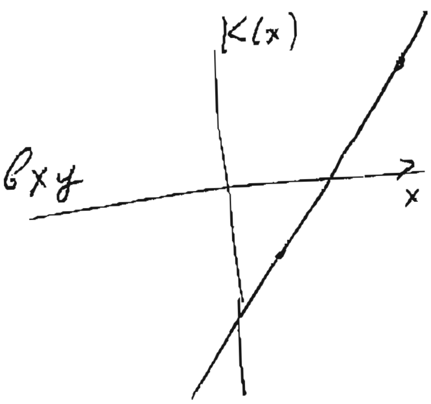
$$\frac{y^2}{2} = -x^2 + \tilde{C}$$

$$\boxed{y^2 + 2x^2 = C}$$ – family of ellipses.

The Lotka-Volterra Model

$x(t)$ – the number of ~~predators~~ prey (rabbits)

$y(t)$ – the number of predators (wolfes)



$$\begin{cases} \frac{d}{dt}x = kx - axy \\ \frac{d}{dt}y = \underbrace{(-\ell + bx)}_{K(x)}y = -\ell y + bxy \end{cases}$$

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{dx}{dy}$$

(Indeed, if $x = x(y(t))$, then $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$

$$\frac{dx}{dy} = \frac{kx - axy}{-\ell y + bxy}$$

$$\frac{dx}{dy} = \frac{x(k-ay)}{y(-l+bx)} = \frac{x}{-l+bx} \cdot \frac{k-ay}{y}$$

$$\int \frac{(-l+bx)}{x}\, dx = \int \frac{k-ay}{y}\, dy + C$$

$$(bx - l \ln x) = (k \ln y - ay) + C$$

$$\underbrace{(bx - l \ln x)}_{p(x)} + \underbrace{(ay - k \ln y)}_{q(y)} = C$$