Introduction to differential equations

Laplace transform

Def

$f(t),\ t \geq 0$ - piecewise continuous, of exponential order,

$$\mathcal{L}(f(t)) = F(s) \equiv \int_0^\infty e^{-ts} f(t)\,dt$$

Properties of Laplace transform

1) $\mathcal{L}(1) = \frac{1}{s}, \quad s > 0$

2) $\mathcal{L}(e^{ta}) = \frac{1}{s-a}, \quad s > a$

3) $\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}, \quad s > 0$

4) $\mathcal{L}(a_1 f_1(t) + a_2 f_2(t)) = a_1 \mathcal{L}(f_1(t)) + a_2 \mathcal{L}(f_2(t))$

5) $F(s) = \mathcal{L}(f(t)) \Rightarrow$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2 F(s) - sf(0) - f'(0)$$

6) $F(s) = \mathcal{L}(f(t)) \Rightarrow$

$$\mathcal{L}(-t f(t)) = \frac{d}{ds} F(s)$$

7) $ay'' + by' + cy = f(t), \quad \mathcal{L}(f(t)) = F(s), \quad \mathcal{L}(y(t)) = Y(s),$
$y(0) = y_0,\ y'(0) = y_0'$

$$Y(s) = \frac{(as+b)y_0 + ay_0' + F(s)}{as^2 + bs + c}$$

Proposition

If $\mathcal{L}(f(t)) = F(s)$ then
$$\mathcal{L}(e^{at} f(t)) = F(s-a)$$

Proof

$$\mathcal{L}(e^{at} f(t)) = \int_0^\infty e^{-ts} e^{at} f(t)\,dt =$$
$$= \int_0^\infty e^{(a-s)t} f(t)\,dt = \int_0^\infty e^{-(s-a)} f(t)\,dt \equiv F(s-a) \quad \square$$

Examples

1) $\mathcal{L}(e^{3t} \sin t) = ?$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}, \quad \mathcal{L}(e^{3t} \sin t) = \frac{1}{(s-3)^2+1}$$

2) Assume that $\mathcal{L}(f(t)) = \dfrac{s-11}{49+(s-11)^2}$; $f(t) = ?$

$$\mathcal{L}(\cos 7t) = \frac{s}{49+s^2}, \text{ so}$$
$$\mathcal{L}(e^{11t} \cos 7t) = \frac{s-11}{49+(s-11)^2}, \text{ so } f(t) = e^{11t} \cos 7t.$$

3) $\mathcal{L}(f(t)) = \dfrac{1}{s^2-2s+4}$, $f(t) = ?$

$$\mathcal{L}(f(t)) = \frac{1}{(s-1)^2+3}, \text{ so } \underline{f(t) = \frac{e^t}{\sqrt{3}} \sin(\sqrt{3}\,t)}$$

$$\mathcal{L}(\sin \sqrt{3}\,t) = \frac{\sqrt{3}}{3+s^2} \Rightarrow \mathcal{L}\left(\frac{1}{\sqrt{3}} \sin \sqrt{3}\,t\right) = \frac{1}{3+s^2}$$

4) $y'' - 2y' + 7y = \sin t$, $y(0) = 0$, $y'(0) = 0$

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$$s^2 Y(s) - 2 s Y(s) + 7 Y(s) = \frac{1}{1+s^2}$$

$$Y(s) = \frac{1}{(1+s^2)(s^2-2s+7)} = \frac{1}{(1+s^2)((s-1)^2+6)} =$$

$$= \frac{s^2-2s+7 - (1+s^2)}{(6-2s)(1+s^2)(s^2-2s+7)} =$$

$$= \frac{1}{(6-2s)(1+s^2)} - \frac{1}{(6-2s)(s^2-2s+7)}$$

$$= \frac{As+B}{1+s^2} + \frac{Cs+D}{s^2-2s+7}$$

$$(As+B)(s^2-2s+7) + (1+s^2)(Cs+D) = 1$$

$$(A+C)s^3 + (-2A+B+D)s^2 +$$

$$+ (7A-2B+C)s + (7B+D) = 1$$

$$\begin{cases} A+C=0 & \Rightarrow C=-A \\ -2A+B+D=0 & \Rightarrow D+A=0, \quad D=C=-A \\ 7A-2B+C=0 & \Rightarrow 6A-2B=0, \quad \underline{3A=B} \\ 7B+D=1 & \quad 21A-A=1 \end{cases}$$

$$A = \frac{1}{20}$$

$$B = \frac{3}{20}$$

$$D = C = -\frac{1}{20}$$

$$Y(s) = \frac{s+3}{20(1+s^2)} - \frac{s+1}{20(s^2-2s+7)} =$$

$$= \frac{1}{20}\left(\frac{s}{1+s^2} + \frac{3}{1+s^2} - \frac{s-1}{(s-1)^2+6} - \frac{1}{(s-1)^2+6}\right) =$$

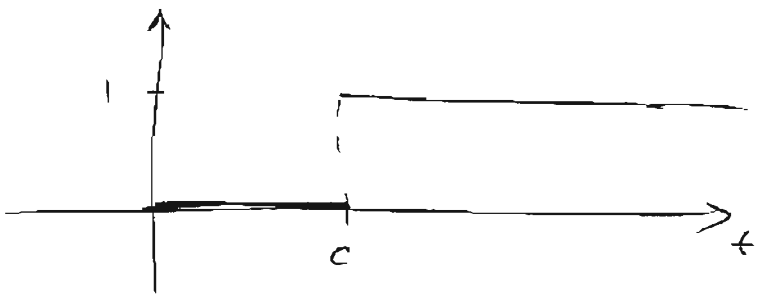
$$= \frac{1}{20}\,\mathcal{L}\Big(\cos t + 3\sin t - e^{t}\cdot \cos\sqrt{6}\,t - $$
$$- e^{t}\frac{1}{\sqrt{6}}\sin\sqrt{6}\,t\Big),$$

so $y(t) = \frac{1}{20}\cos t + \frac{3}{20}\sin t - \frac{1}{20}e^{t}\cos\sqrt{6}\,t - \frac{1}{20}\frac{e^{t}}{\sqrt{6}}\sin\sqrt{6}\,t$

Def

Heaviside function

$$H_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$$



$$\mathcal{L}(H_c(t)) = \int_0^\infty e^{-st} H_c(t)\,dt = \int_c^\infty e^{-st}\,dt =$$
$$= \lim_{A\to\infty}\int_c^A e^{-st}\,dt = \lim_{A\to\infty}\frac{e^{-cs} - e^{-sA}}{s} = \frac{e^{-cs}}{s}, \quad s > 0$$

Proposition

If $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}(H_c(t)\cdot f(t-c)) = e^{-cs}F(s)$$

Proof

$$\mathcal{L}\left(H_c(t) f(t-c)\right) = \int_0^\infty e^{-st} H_c(t) f(t-c)\,dt =$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt = \| \xi = t-c \| = \int_0^\infty e^{-s(\xi+c)} f(\xi)\,d\xi =$$

$$= e^{-sc} \int_0^\infty e^{-s\xi} f(\xi)\,d\xi = e^{-cs} F(s) \qquad \square$$

Example

1) $Y(s) = \dfrac{e^{-s}}{s^2}$, $y(t) = ?$

$$\mathcal{L}(t) = \frac{1}{s^2}, \text{ so } \frac{e^{-s}}{s^2} = \mathcal{L}\left(H_1(t)(t-1)\right),$$

$$y(t) = H_1(t)(t-1).$$

2)

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & 3 \le t \end{cases} \qquad F(s) = ?$$

$$f(t) = \left(H_0(t) - H_1(t)\right) + \left(H_2(t) - H_3(t)\right)$$

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} =$$

$$= \frac{1 - e^{s} + e^{-2s} - e^{-3s}}{s}, \quad s > 0$$

Example

$$y'' + 4y = f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & t > 4 \end{cases} \qquad y(0) = 3,\ y'(0) = -2$$

$$\mathcal{L}(f(t)) = \mathcal{L}(H_0(t) - H_4(t)) = \frac{1}{s} - \frac{e^{-4s}}{s}$$

$$s^2 Y(s) - s \cdot 3 + 2 + 4Y(s) = \frac{1 - e^{-4s}}{s}$$

$$Y(s) = \frac{1}{s^2+4}\left(\frac{1}{s} + 3s - 2\right) - e^{-4s}\frac{1}{s(s^2+4)} =$$

$$= 3\frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{1}{s(s^2+4)} - e^{-4s}\frac{1}{s(s^2+4)} =$$

$$= 3\frac{s}{s^2+4} - \frac{2}{s^2+4} + (1 - e^{-4s})\left(\frac{s^2+4-s^2}{4s(s^2+4)}\right) =$$

$$= 3\frac{s}{s^2+4} - \frac{2}{s^2+4} + (1 - e^{-4s})\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) =$$

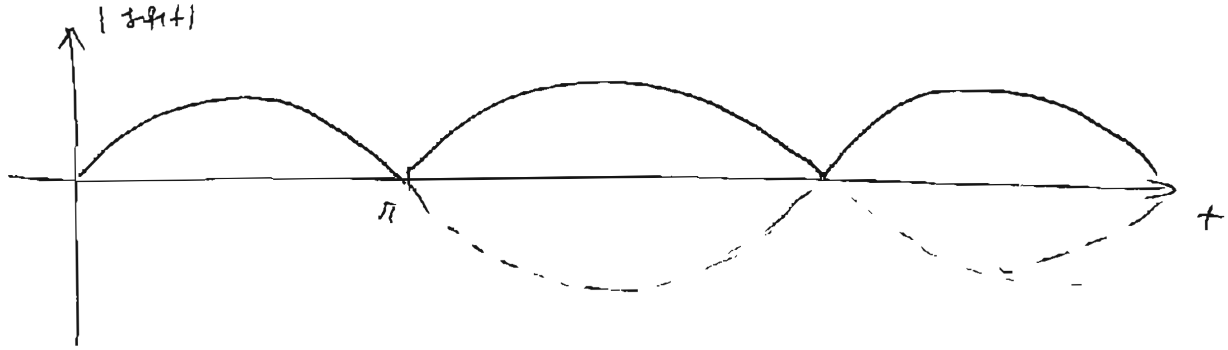
$$= \mathcal{L}\left(3\cos 2t - \sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - H_4(t)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-4)\right)\right)$$

so
$$y(t) = 3\cos 2t - \sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4} H_4(t) \cdot (1 - \cos 2(t-4))$$

Remark Both $y(t)$ and $y'(t)$ are continuous for $t \geq 0$!

Example

$$f(t) = |\sin t| = \sin t + 2\sum_{n=1}^{\infty} H_{n\pi}(t)\sin(t-n\pi)$$



$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} + 2\sum_{n=1}^{\infty} e^{-n\pi s}\cdot\frac{1}{1+s^2} =$$

$$= \frac{1}{1+s^2}\left(1 + 2\sum_{n=1}^{\infty} e^{-n\pi s}\right) =$$

$$= \frac{1}{1+s^2}\left(1 + 2\,\frac{e^{-\pi s}}{1-e^{-\pi s}}\right) =$$

$$= \frac{1}{1+s^2}\cdot\frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$