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Introduction to differential equations

Laplace transform

Def

Let $f(t)$, $t \geq 0$, be a piecewise continuous function of exponential order (i.e. $\exists\, M > 0$, $c > 0$ s.t. $|f(t)| \leq M e^{ct}$ $\forall t \geq 0$). Then the Laplace transform of $f(t)$ is a function $F(s) = \mathcal{L}(f(t))$,

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Examples

1) $f(t) \equiv 1$, $\mathcal{L}(f(t)) = \begin{cases} \frac{1}{s}, & s > 0 \\ \infty, & s \leq 0 \end{cases}$

2) $\mathcal{L}(e^{at}) = \begin{cases} \frac{1}{s-a}, & s > a \\ \infty, & s \leq a \end{cases}$

3) $\mathcal{L}(\cos \omega t) = \dfrac{s}{s^2 + \omega^2}$, $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2 + \omega^2}$, $s > 0$

Remark

$\mathcal{L}$ is a linear operator:

$$\mathcal{L}(a_1 f_1(t) + a_2 f_2(t)) = a_1 \mathcal{L}(f_1(t)) + a_2 \mathcal{L}(f_2(t))$$

Lemma

Let $F(s) = \mathcal{L}(f(t))$. Then

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$$

How to solve a linear differential equation using Laplace transform:

$$(*) \quad a y'' + b y' + c y = f(t) \quad , \quad y(0) = y_0 \, , \; y'(0) = y_0'$$

$$\mathcal{L}(y(t)) = Y(s), \quad \mathcal{L}(f(t)) = F(s)$$

$$\mathcal{L}(ay'' + by' + cy) = \mathcal{L}(f(t))$$

$$a\,\mathcal{L}(y''(t)) + b\,\mathcal{L}(y'(t)) + c\,\mathcal{L}(y(t)) = \mathcal{L}(f(t))$$

$$a(s^2 Y(s) - s y_0 - y_0') + b(sY(s) - y_0) + cY(s) = F(s)$$

$$Y(s) = \frac{(as+b)y_0}{as^2+bs+c} + \frac{a y_0'}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$$

If $y(t)$ is such that $\mathcal{L}(y(t)) = Y(s)$ (i.e. $y(t) = \mathcal{L}^{-1}(Y(s))$) then $y(t)$ is the solution of the initial value problem $(*)$.

Example

$$y'' - 3y' + 2y = e^{3t} \qquad y(0) = 0, \; y'(0) = 0$$

Solution

$$\mathcal{L}(y'' - 3y' + 2y) = \mathcal{L}(e^{3t})$$

$$\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y) = \frac{1}{s-3}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - 3(s\,Y(s) - y(0)) + 2\,Y(s) = \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s-3)(s^2-3s+2)} = \frac{1}{(s-3)(s-2)(s-1)} =$$

$$= \frac{1}{2(s-3)} + \frac{1}{2(s-1)} - \frac{1}{s-2},$$

$$Y(s) = \mathcal{L}\left(\tfrac{1}{2}e^{3t} + \tfrac{1}{2}e^{t} - e^{2t}\right), \quad \text{so}$$

$y(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{t} - e^{2t}$ is a solution of the given initial value problem.

Proposition 1

If $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}(-t f(t)) = \frac{d}{ds} F(s)$$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt =$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt = \mathcal{L}(-t f(t)) \quad \square$$

Examples

1) $$\mathcal{L}(te^t) = -\frac{d}{ds}\mathcal{L}(e^t) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

2) $$\mathcal{L}(t^{17}) = (-1)^{17}\frac{d^{17}}{ds^{17}}\mathcal{L}(1) = (-1)^{17}\frac{d^{17}}{ds^{17}}\frac{1}{s} = \frac{(17)!}{s^{18}}$$

3) $$\mathcal{L}(y(t)) = Y(s) = -\frac{1}{(s-2)^2}$$

$$-\frac{1}{(s-2)^2} = \frac{d}{ds}\frac{1}{s-2}, \qquad \frac{1}{s-2} = \mathcal{L}(e^{2t}), \text{ so}$$

$$\mathcal{L}^{-1}\left(-\frac{1}{(s-2)^2}\right) = -te^{2t}$$

4) $Y(s) = \mathcal{L}(y(t))$, $Y(s) = -\dfrac{4s}{(s^2+4)^2}$

$$-\frac{4s}{(s^2+4)^2} = \frac{d}{ds}\,\frac{2}{s^2+4}\,, \quad \frac{2}{s^2+4} = \mathcal{L}(\sin 2t)$$

$$\mathcal{L}^{-1}(Y(s)) = y(t) = -t \sin 2t$$

5)

$$Y(s) = \frac{1}{(s-4)^3}$$

$$\frac{1}{(s-4)^3} = \frac{d^2}{ds^2}\,\frac{1}{2}\,\frac{1}{s-4}\,, \quad \text{so } \frac{1}{(s-4)^3} = \mathcal{L}\left(\frac{1}{2}t^2e^{4t}\right),$$

$$y(t) = \frac{1}{2}t^2e^{4t}$$

<u>Proposition 2</u>

If $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

<u>Proof</u>

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{(a-s)t}f(t)\,dt =$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt \equiv F(s-a) \qquad \square$$

Examples

1) $\mathcal{L}(e^{5t}\sin t) = \dfrac{1}{(s-5)^2+1}$ (since $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$)

2) $Y(s) = \dfrac{s-7}{25+(s-7)^2}$

Notice that $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+25}\right) = \cos 5t$,

and therefore $\mathcal{L}^{-1}(Y(s)) = e^{7t}\cos 5t$.

3) $Y(s) = \dfrac{1}{s^2-4s+9} = \dfrac{1}{(s-2)^2+5}$

$\mathcal{L}\left(\dfrac{1}{\sqrt{5}}\sin\sqrt{5}\,t\right) = \dfrac{1}{s^2+5}$, so

$$y(t) = \frac{e^{2t}}{\sqrt{5}}\sin\sqrt{5}\,t.$$

Example

$$y'' + y = \cos t, \quad y(0) = y'(0) = 0$$

$$\mathcal{L}(y''+y) = \mathcal{L}(\cos t)$$

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s^2+1)^2}$$

$\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right) = -\dfrac{2s}{(s^2+1)^2}$, so $\dfrac{d}{ds}\left(-\dfrac{1}{2}\cdot\dfrac{1}{1+s^2}\right) = \dfrac{s}{(s^2+1)^2}$

$\mathcal{L}\left(-\dfrac{1}{2}\sin t\right) = -\dfrac{1}{2}\,\dfrac{1}{1+s^2}$, so $y(t) = \dfrac{1}{2}\,t\sin t$ is the solution.