# Introduction to differential equations

Linear homogeneous system of first order differential equations with constant coefficients

$$(*) \quad \dot{\bar{x}} = A\bar{x}, \quad \bar{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

We know that if $\bar{x}_1, \dots, \bar{x}_n$ are $n$ linearly independent solutions then a general solution has the form

$$\bar{x}(t) = C_1\bar{x}_1 + C_2\bar{x}_2 + \dots + C_n\bar{x}_n.$$

Let us try to find a solution of $(*)$ in the form $\bar{x}(t) = e^{\lambda t}\cdot\bar{v}$, where $\bar{v}$ is some constant vector.

$$\frac{d}{dt}\bar{x}(t) = \frac{d}{dt}(e^{\lambda t}\bar{v}) = \lambda e^{\lambda t}\bar{v},$$

$A(e^{\lambda t}\bar{v}) = e^{\lambda t}A\bar{v}$, so if $\frac{d}{dt}\bar{x} = A\bar{x}$ then

$$\lambda e^{\lambda t}\bar{v} = e^{\lambda t}A\bar{v},$$

$$\boxed{A\bar{v} = \lambda\bar{v}}$$

Def.

A vector $\bar{v} \neq 0$ is an eigenvector of $A$ if $A\bar{v} = \lambda \bar{v}$; $\lambda$ is an eigenvalue that corresponds to $\bar{v}$.

If $\bar{v}$ is an eigenvector of $A$ then

$e^{\lambda t}\bar{v}$ is a solution of (*).

How to find $\bar{v}$?

If $A\bar{v} = \lambda\bar{v}$ then

$$A\bar{v} - \lambda\bar{v} = \bar{0}$$

$$(A - \lambda I)\bar{v} = 0, \quad I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$$

This equation has a non-zero solution iff

$\underline{\det(A - \lambda I) = 0}$

This is a polynomial of degree $n$ in $\lambda$.

It has at least one and at most $n$ different (possibly complex) roots.

If all roots are different and real then each of them corresponds to an eigenvector:

$\lambda_1, \ldots, \lambda_n$, $\bar{v}_1, \ldots, \bar{v}_n$.

Thm

Any $k$ eigenvectors $\bar{v}_1, \ldots, \bar{v}_k$ of $A$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ are linearly independent.

Example

$$\frac{d}{dt}\bar{x} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \bar{x} \quad , \quad \bar{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$$

Find eigenvalues of $A$:

$$\det \begin{pmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{pmatrix} = 0$$

$$(6-\lambda)(1-\lambda) + 6 = 0$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$\lambda = \frac{7 \pm \sqrt{49-48}}{2} = \{4, 3\}$$

Find eigenvector for $\lambda = 3$:

$$(A - \lambda_1 I)\bar{v} = 0 \quad , \quad \bar{v}' = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can take $\bar{v}' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

## Find eigenvector for $\lambda = 4$



$$(A - \lambda_2 I)\,\bar{v}^2 = 0 \qquad \bar{v}^2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can take $\bar{v}^2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Therefore a general solution is

$$\bar{x}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 e^{3t} + 3C_2 e^{4t} \\ C_1 e^{3t} + 2C_2 e^{4t} \end{pmatrix} \qquad \square$$

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**Example 2** Solve IVP:

$$\frac{d}{dt}\bar{x} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\bar{x}, \quad \bar{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0 = \lambda^2 - 3\lambda - 4$,

$\lambda_1 = -1,\ \lambda_2 = 4$

Eigenvectors: $\lambda_1 = -1$

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ take } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 4$

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{take } \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$\bar{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$ - general solution

$$\bar{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

so $$\begin{cases} -c_1 + 2c_2 = 0 \\ c_1 + 3c_2 = -4 \end{cases} \Rightarrow c_1 = -\frac{8}{5}, \; c_2 = -\frac{4}{5},$$

so the solution of IVP is

$$\bar{x}(t) = -\frac{8}{5} e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5} e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

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## Complex roots

$$(*) \qquad \frac{d}{dt}\bar{x} = A x$$

If $\lambda = \alpha + i\beta$ is an eigenvalue, and $\bar{v} = \bar{v}_1 + i\bar{v}_2$ is an eigenvector then $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue, and

$\bar{x}(t) = e^{\lambda t}\bar{v}$ is a complex-valued solution of $(*)$.

## Lemma

If $\bar{x}(t) = \bar{y}(t) + i\,\bar{z}(t)$ is a complex-valued solution of (*) then $\bar{y}(t)$ and $\bar{z}(t)$ are real-valued solutions.

$$\bar{x}(t) = e^{(\alpha + i\beta)t}(\bar{v}_1 + i\bar{v}_2) =$$

$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\bar{v}_1 + i\bar{v}_2) =$$

$$= e^{\alpha t}\left[(\bar{v}_1\cos\beta t - \bar{v}_2\sin\beta t) + i(\bar{v}_1\sin\beta t + \bar{v}_2\cos\beta t)\right],$$

therefore $\operatorname{Re}\bar{x}(t) = e^{\alpha t}(\bar{v}_1\cos\beta t - \bar{v}_2\sin\beta t)$,

$\operatorname{Im}\bar{x}(t) = e^{\alpha t}(\bar{v}_1\sin\beta t + v_2\cos\beta t)$

are solutions (linearly independent!).

## Example

$$\frac{d}{dt}\bar{x}(t) = A\,x(t), \qquad A = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}$$

$$\det(A - \lambda I) = 0 = \lambda^2 + 27,$$

$$\lambda_{1,2} = \pm 3\sqrt{3}\,i \qquad (\lambda = \alpha + i\beta,\ \alpha = 0,\ \beta = 3\sqrt{3})$$

Eigenvector:

$$\begin{pmatrix} 3 - 3\sqrt{3}\,i & -9 \\ 4 & -3 - 3\sqrt{3}\,i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can take

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$$\bar{v} = \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} i$$

$$\bar{X}(t) = e^{3\sqrt{3}t i} \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix} = \left(\cos 3\sqrt{3}t + i \sin 3\sqrt{3}t\right) \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos 3\sqrt{3}t + i\, 3 \sin 3\sqrt{3}t \\ \left(\cos 3\sqrt{3}t + \sqrt{3}\sin 3\sqrt{3}t\right) + i\left(\sin 3\sqrt{3}t - \sqrt{3}\cos 3\sqrt{3}t\right) \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ \cos 3\sqrt{3}t + \sqrt{3}\sin 3\sqrt{3}t \end{pmatrix} + i \begin{pmatrix} 3\sin 3\sqrt{3}t \\ \sin 3\sqrt{3}t - \sqrt{3}\cos 3\sqrt{3}t \end{pmatrix},$$

Therefore the general solution has the form

$$\bar{X}(t) = C_1 \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + \sqrt{3}\sin 3\sqrt{3}t \end{pmatrix} +$$

$$+ C_2 \begin{pmatrix} 3\sin(3\sqrt{3}t) \\ \sin 3\sqrt{3}t - \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}$$

## Example 2

$$\frac{d}{dt}\bar{x} = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\bar{x}$$

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 68, \quad \lambda_{1,2} = 2 \pm 8i$$

Eigenvector (for $\lambda = 2+8i$)

$$\begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can take $\bar{v} = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$,

$$\bar{x}(t) = e^{(2+8i)t}\begin{pmatrix} 1+8i \\ 5 \end{pmatrix} = e^{2t}(\cos(8t) + i\sin(8t))\begin{pmatrix} 1+8i \\ 5 \end{pmatrix} =$$

$$= e^{2t}\begin{pmatrix} \cos 8t - 8\sin 8t \\ 5\cos 8t \end{pmatrix} + i e^{2t}\begin{pmatrix} 8\cos 8t + \sin 8t \\ 5\sin 8t \end{pmatrix},$$

so the general solution is

$$\bar{x}(t) = c_1 e^{2t}\begin{pmatrix} \cos 8t - 8\sin 8t \\ 5\cos 8t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 8\cos 8t + \sin 8t \\ 5\sin 8t \end{pmatrix}$$