Properties of solutions

Assume that $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ are solutions,

$$\begin{cases} \dot{x}_1 = a_{11} x_1 + \dots + a_{1n} x_n \\ \vdots \\ \dot{x}_n = a_{n1} x_1 + \dots + a_{nn} x_n \end{cases}, \qquad \begin{cases} \dot{y}_1 = a_{11} y_1 + \dots + a_{1n} y_n \\ \vdots \\ \dot{y}_n = a_{n1} y_1 + \dots + a_{nn} y_n, \end{cases}$$

then

$$\begin{cases} \frac{d}{dt}(x_1 + y_1) = a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2) + \dots + a_{1n}(x_n + y_n) \\ \vdots \\ \frac{d}{dt}(x_n + y_n) = a_{n1}(x_1 + y_1) + \dots + a_{nn}(x_n + y_n), \end{cases}$$

so $\bar{x} + \bar{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$ is also a solution.

Also, $\forall c \in \mathbb{R}$ $\quad c\bar{x} = \begin{pmatrix} c x_1 \\ \vdots \\ c x_n \end{pmatrix}$ is a solution

(i) $\quad \bar{x} + \bar{y} = \bar{y} + \bar{x}$

(ii) $\quad \bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$

(iii) $\quad \bar{0}(t) \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution, and if $\bar{x}$ is a solution, then $\bar{x} + \bar{0} = \bar{0} + \bar{x} = \bar{x}$.

(iv) $\quad$ If $\bar{x}$ is a solution then $-\bar{x} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$ is a solution, and $\bar{x} + (-\bar{x}) \equiv \bar{0}$

(v) $\quad 1 \cdot \bar{x} = \bar{x}$

(vi) $\quad$ For any numbers $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$) $\quad (ab)\bar{x} = a \cdot (b\bar{x})$

(vii) $a(\bar{x}+\bar{y}) = a\bar{x} + a\bar{y}$

(viii) $(a+b)\bar{x} = a\bar{x} + b\bar{x}$

Def

A set L of elements called „vectors" $(\bar{x}, \bar{y}, \bar{z}, \ldots)$ is a linear space if vectors can be multiplied by a number (so that $\forall a \in \mathbb{R}$ $a\bar{x}$ is also a vector), and an addition of vectors is defined (so $\bar{x}+\bar{y}$ is also a vector), and properties (i) - (viii) above hold.

Examples

1) Consider $\mathbb{R}^2 = \{(x,y) \mid, x, y \in \mathbb{R}\}$, where

$$(x,y) + (z,w) = (x+z, y+w), \quad a(x,y) = (ax, ay).$$

2) $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_i \in \mathbb{R}$,

$$\bar{x}+\bar{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}, \quad a\bar{x} = \begin{pmatrix} ax_1 \\ \vdots \\ ax_n \end{pmatrix}.$$

4) Consider all continuous functions

$\bar{x} = x(t)$, $t \in [0,1]$, $\bar{x}+\bar{y} = x(t)+y(t) : [0,1] \to \mathbb{R}$,

$$a\bar{x} = ax(t) : [0,1] \to \mathbb{R}$$

$\bar{0} \equiv 0$

5) $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_1 y' + a_0 y = 0$

The space of all solutions $y(t)$ is a vector space;

$y_1(t)$, $y_2(t)$ – are solutions $\Rightarrow$

$y_1 + y_2$ is a solution, αy_1 is a solution.

6) Let L be the space of all matrices 2×2,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ b_{21} + a_{21} & a_{22} + b_{22} \end{pmatrix}, \quad cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

7) $\dot{\bar{x}} = A(t)\,\bar{x}$,

the space of solutions $\bar{x}(t)$ is a vector space.

<u>Def</u>

A set of vectors $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ is said to span a linear space L if any vector $\bar{v} \in L$ can be represented as

$$\bar{v} = c_1 \bar{x}^1 + c_2 \bar{x}^2 + \dots + c_n \bar{x}^n, \text{ for some numbers } c_i \in \mathbb{R}.$$

Def

The dimension of a vector space L ($\dim L$) is the fewest number of linearly independent vectors $\bar{x}^1, \dots, \bar{x}^n$ which span L.

Def

$\bar{x}^1, \dots, \bar{x}^n$ are linearly independent if

$$c_1 \bar{x}^1 + \dots + c_n \bar{x}^n = \bar{0} \Rightarrow c_1 = c_2 = \dots = c_n = 0.$$

Theorem 1 (Existence and uniqueness)

$\dot{\bar{x}} = A \bar{x}$, ~~$A(t)$ is continuous.~~ A is a constant matrix

There exists one, and only one solution of the initial-value problem

$$\frac{d}{dt} \bar{x} = A \bar{x}, \quad \bar{x}(0) = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix},$$

and this solution exists for all $t \in \mathbb{R}$.

Thm 2

All solutions of the system $\dot{\bar{x}} = A x$, $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, form a linear space of dimension n.

The idea of the proof:

Consider solutions $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ with initial conditions $\bar{x}^1(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\bar{x}^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \bar{x}^n(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

These solutions are linearly independent, and if $\bar{y}(t) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a solution, then

Consider $y_1(0) \cdot \bar{x}^1 + y_2(0)\bar{x}^2 + \cdots + y_n(0)\bar{x}^n$.

This is also a solution, and its initial conditions are the same as for $y(t)$, so

$$y(t) = y_1(0)\bar{x}^1 + \cdots + y_n(0)\bar{x}^n. \qquad \square$$