Introduction to differential equations

Systems of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(t, x_1, \ldots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n) \end{cases}$$

System of n first-order differential equations.

Remark

A differential equation of order n can be converted into a system of n first-order differential equations.

Example

$$a_n(t)\frac{d^n y}{dt^n} + \ldots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0$$

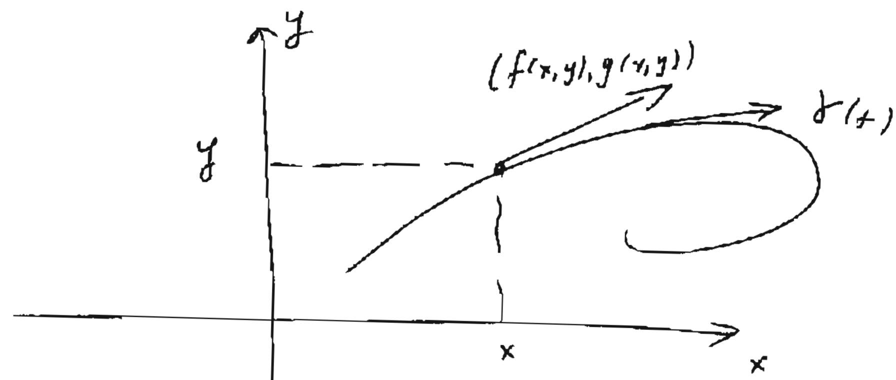
Set $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = y''(t)$, …, $x_n = \dfrac{d^{n-1}y}{dy^{n-1}}$,

then

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = -\dfrac{a_{n-1}(t)x_n + a_{n-2}(t)x_{n-1} + \ldots + a_0(t)x_1}{a_n(t)} \end{cases}$$

Geometrical interpretation

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$



If $(x(t), y(t))$ is a solution

then $(\dot{x}(t), \dot{y}(t)) = (f(x,y), g(x,y))$,

so the tangent vector to the curve

$\gamma(t) = (x(t), y(t))$ at the point $(x(t), y(t))$ is

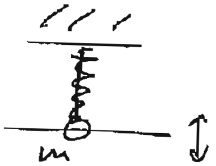
$(f(x,y), g(x,y))$.

Geometrical problem

Given a vector field on $\mathbb{R}^2$, find curves on $\mathbb{R}^2$ such that at each point of the curve its derivative is equal to the vector attached at that point.

Example

Harmonic oscillator



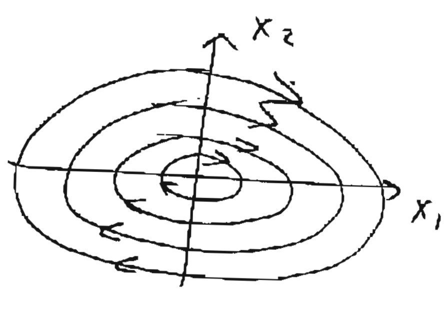
$$m\ddot{x} = -kx$$

$$x(t) = R\cos\left(\sqrt{\tfrac{k}{m}}\, t + \omega\right)$$

$X_1 = x$

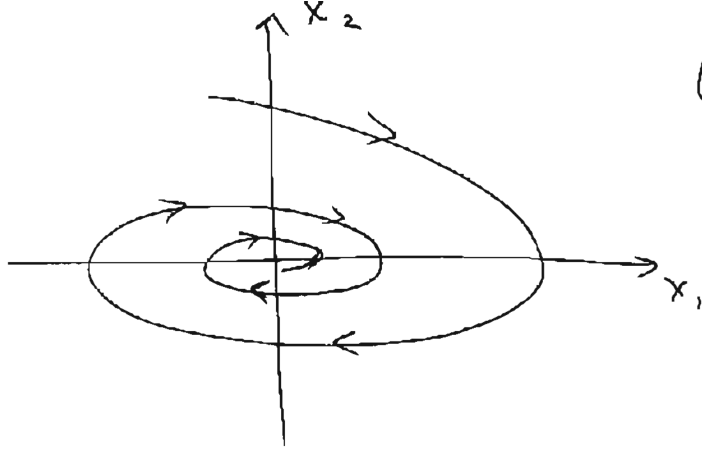
$X_2 = \dot{x}$,

$$\begin{cases} \dot{X}_1 = X_2 \\ \dot{X}_2 = -\frac{k}{m} X_1 \end{cases}$$



$$m\ddot{x} = -kx - c\dot{x}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{cases}$$



Linear homogeneous systems of differential equations of first order.

$$\begin{cases} \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n \\ \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n \\ \vdots \\ \dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n \end{cases}$$

Denote

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \bar{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$ – vector-function.

$$A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}$$ – $n \times n$ matrix

Now we can write the system as

$$\boxed{\dot{\bar{x}} = A(t) \cdot \bar{x}}$$

In the case when $a_{ij}(t)$ are constants,

A is a constant matrix, and

$$\begin{cases} \dot{x}_1 = a_{11} x_1 + \dots + a_{1n} x_n \\ \vdots \\ \dot{x}_n = a_{n1} x_1 + \dots + a_{nn} x_n \end{cases}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

$$\boxed{\dot{\bar{x}} = A \bar{x}}$$

Example

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - x_2 + x_3 \\ \dfrac{dx_2}{dt} = 2x_1 - 3x_2 + 5x_3 \\ \dfrac{dx_3}{dt} = 7x_2 - x_3 \end{cases}, \quad x_1(0) = 5,\ x_2(0) = 0,\ x_3(0) = 3$$

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 5 \\ 0 & 7 & -1 \end{pmatrix},$$

$$\dot{\bar{x}} = A\bar{x}, \quad \bar{x}(0) = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

Properties of solutions

Assume that $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ are solutions,

$$\begin{cases} \dot{x}_1 = a_{11} x_1 + \dots + a_{1n} x_n \\ \vdots \\ \dot{x}_n = a_{n1} x_1 + \dots + a_{nn} x_n \end{cases}, \qquad \begin{cases} \dot{y}_1 = a_{11} y_1 + \dots + a_{1n} y_n \\ \vdots \\ \dot{y}_n = a_{n1} y_1 + \dots + a_{nn} y_n, \end{cases}$$

then
$$\begin{cases} \frac{d}{dt}(x_1 + y_1) = a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2) + \dots + a_{1n}(x_n + y_n) \\ \vdots \\ \frac{d}{dt}(x_n + y_n) = a_{n1}(x_1 + y_1) + \dots + a_{nn}(x_n + y_n), \end{cases}$$

so $\bar{x} + \bar{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$ is also a solution.

Also, $\forall c \in \mathbb{R}$ $\quad c\bar{x} = \begin{pmatrix} c x_1 \\ \vdots \\ c x_n \end{pmatrix}$ is a solution

(i) $\bar{x} + \bar{y} = \bar{y} + \bar{x}$

(ii) $\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$

(iii) $\bar{0}(t) \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution, and if $\bar{x}$ is a solution, then $\bar{x} + \bar{0} = \bar{0} + \bar{x} = \bar{x}$.

(iv) If $\bar{x}$ is a solution then $-\bar{x} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$ is a solution, and $\bar{x} + (-\bar{x}) \equiv \bar{0}$

(v) $1 \cdot \bar{x} = \bar{x}$

(vi) For any numbers $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$) $(ab)\bar{x} = a \cdot (b\bar{x})$

(vii) $a(\bar{x}+\bar{y}) = a\bar{x} + a\bar{y}$

(viii) $(a+b)\bar{x} = a\bar{x} + b\bar{x}$

Def

A set L of elements called „vectors" $(\bar{x}, \bar{y}, \bar{z}, \ldots)$ is a linear space if vectors can be multiplied by a number (so that $\forall a \in \mathbb{R}$ $a\bar{x}$ is also a vector), and an addition of vectors is defined (so $\bar{x}+\bar{y}$ is also a vector), and properties (i) - (viii) above hold.

Examples

1) Consider $\mathbb{R}^2 = \{(x,y) \mid, x, y \in \mathbb{R}\}$, where

$$(x,y) + (z,w) = (x+z, y+w), \quad a(x,y) = (ax, ay).$$

2) $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_i \in \mathbb{R}$,

$$\bar{x}+\bar{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}, \quad a\bar{x} = \begin{pmatrix} ax_1 \\ \vdots \\ ax_n \end{pmatrix}.$$

4) Consider all continuous functions

$\bar{x} = x(t)$, $t \in [0,1]$, $\bar{x}+\bar{y} = x(t)+y(t) : [0,1] \to \mathbb{R}$,

$$a\bar{x} = ax(t) : [0,1] \to \mathbb{R}$$

$\bar{0} \equiv 0$

5) $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_1 y' + a_0 y = 0$

The space of all solutions $y(t)$ is a vector space:

$y_1(t)$, $y_2(t)$ – are solutions $\Rightarrow$

$y_1 + y_2$ is a solution, αy_1 is a solution.

6) Let L be the space of all matrices 2×2,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ b_{21} + a_{21} & a_{22} + b_{22} \end{pmatrix}, \quad cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

7) $\dot{\bar{x}} = A(t)\,\bar{x}$,

the space of solutions $\bar{x}(t)$ is a vector space.

Def

A set of vectors $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ is said to span a linear space L if any vector $\bar{v} \in L$ can be represented as

$$\bar{v} = c_1 \bar{x}^1 + c_2 \bar{x}^2 + \dots + c_n \bar{x}^n, \text{ for some numbers } c_i \in \mathbb{R}.$$

Def

The dimension of a vector space L ($\dim L$) is the fewest number of linearly independent vectors $\bar{x}^1, \ldots, \bar{x}^n$ which span L.

Def

$\bar{x}^1, \ldots, \bar{x}^n$ are linearly independent if

$$c_1 \bar{x}^1 + \cdots + c_n \bar{x}^n = \bar{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0.$$

Theorem 1 (Existence and uniqueness)

$\dot{\bar{x}} = A \bar{x}$, ~~A(t) is continuous.~~ A is a constant matrix

There exists one, and only one solution of the initial-value problem

$$\frac{d}{dt}\bar{x} = A\bar{x}, \quad \bar{x}(0) = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix},$$

and this solution exists for all $t \in \mathbb{R}$.

Thm 2

All solutions of the system $\dot{\bar{x}} = A x$, $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, form a linear space of dimension n.

The idea of the proof:

Consider solutions $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ with initial conditions $\bar{x}^1(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\bar{x}^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $\bar{x}^n(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

These solutions are linearly independent, ⑨
and if $\bar{y}(t) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a solution, then

Consider $y_1(0) \cdot \bar{x}^1 + y_2(0)\,\bar{x}^2 + \cdots + y_n(0)\,\bar{x}^n$.

This is also a solution, and its initial conditions are the same as for $y(t)$, so

$$y(t) = y_1(0)\,\bar{x}^1 + \cdots + y_n(0)\,\bar{x}^n. \qquad \square$$