Introduction to partial differential equations – The heat equation; separation of variables.

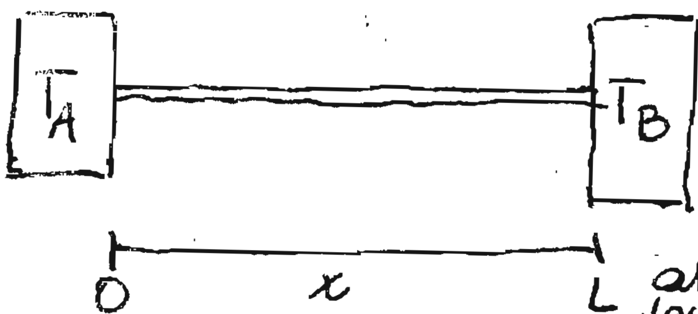
Chapter 5, p.481 f.

1. Introduction

So far considered ordinary differential equs. (ODE), i.e. the unknown fcn. depends on only 1 variable. Now want to consider partial differential equs. (PDE), where the unknown fcn. depends on more than one variable. A partial dif. eqn. is ~~…~~ a relation that involves this fcn and some of its partial derivatives.

examples: "classical" PDEs

① heat flux between two bodies:
consider two bodies at fixed temperatures T_A and T_B, $T_A < T_B$. Connect them by a thin metal slab.



$\Rightarrow$ heat exchange from $A \to B$
$\Rightarrow$ want to know about temperature distribution along the slab

$T(x,t)$ ... temp. at position x in the slab and time t

$T(x,t)$ is governed by heat equation:

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \qquad (1)$$

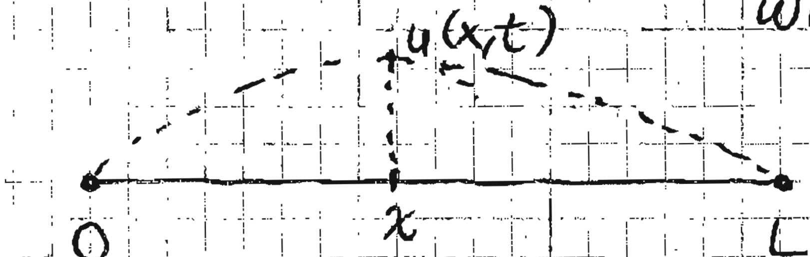
K ... thermal conductivity ($K>0$).

<u>well posed problem:</u>

solve (1) given
- (i) <u>initial</u> temperature distrib. ($t=0$) $T(x,0)$
- (ii) <u>boundary condition:</u> $T(0,t) = T_A$, $T(L,t) = T_B$, all t.

② <u>wave equation</u>

Consider a string fixed at its endpoints. (eg. string of a guitar). Describe "profile" of string at time t by $u(x,t)$:



When string is plugged, $u(x,t)$ will change.

This is described by wave equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (2)$$

c ... speed of propagation

Solve (2) given that:
- (i) initial profile $u(x,0)$
- (ii) initial speed $u_t(x,0)$, where $u_t = \frac{\partial u}{\partial t}$

(iii) boundary condition:

$$u(0,t) = u(L,t) = 0 \quad \text{(string is fixed)}.$$

③ <u>Laplace equation</u>

$$\begin{cases} \Delta u = 0, \text{ where} \\ \Delta := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \quad (\text{in } \mathbb{R}^2) \end{cases} \qquad (3)$$

... Laplace operator

application in electrostatics where u describes the electrostatic potential in vacuum.

Want to focus on the heat equation (1).

<u>terminology:</u> Similarly to ODE we <u>define the order of a PDE</u> to be the <u>order</u> of the highest partial derivative involved in the eqn. ~~Eg. the heat eqn~~
Above named examples are all of order two.

<u>2. The heat equation - separation of variables</u>

Want to solve the following <u>boundary value problem</u> (BVP),

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \end{cases} \qquad (4)$$

(ii)

~~Note that (?) involves initial and boundary conditions (Bc): Subject to the initial condition~~

properties of solutions to (4):

Prop.: Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions to the BVP in (4), then so is any linear combination, $(c_1 u_1 + c_2 u_2)$ $c_1, c_2 \in \mathbb{R}$.

Proof: Since both u_1, u_2 are slns to 4, we have for $c_1, c_2 \in \mathbb{R}$:

$$\left(\alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)(c_1 u_1(x,t) + c_2 u_2(x,t)) =$$

$$= c_1 \underbrace{\left\{\alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right\} u_1(x,t)}_{=0} + c_2 \underbrace{\left\{\alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right\} u_2(x,t)}_{=0}$$

$= 0$; thus $(c_1 u_1(x,t) + c_2 u_2(x,t))$ satisfies the heat equ. Also,

$$c_1 \underbrace{u_1(0,t)}_{=0} + c_2 \underbrace{u_2(0,t)}_{=0} = 0$$

$$c_1 u_1(L,t) + c_2 u_2(L,t) = 0 \text{ , hence}$$

$(c_1 u_1 + c_2 u_2)$ also fulfilles the bc. $\square$

Hence the family of slns. to (4) ("general sln. of the BVP") form a <u>vector space</u>. Additionally we may impose an initial condition on the solutions to (4):

$$u(x, 0) = f(x) \qquad (5),.$$

~~...~~ In our considerations (and in many applications) $f(x)$ is a <u>piecewise continuous fcn.</u> on $[0, L]$.

How to solve (4)?

- <u>Ansatz</u> ("separation of variables"):

  write $u(x,t) = X(x) \cdot T(t)$ (6)

  (6) is a sln to (4) iff ~~...~~

$$\left[\alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right] u(x,t) = \left[\alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right] X(x)\,T(t) =$$

$$= \alpha^2 \frac{\partial^2}{\partial x^2}\left(X(x)T(t)\right) - \frac{\partial}{\partial t}\left(X(x)T(t)\right) =$$

$$= \alpha^2 T(t)\,X''(x) - X(x)\,T'(t) = 0$$

$$\Longleftrightarrow \quad \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} \quad , \text{ \underline{all} } x \text{ and } t$$

$$\Longleftrightarrow \quad \frac{X''(x)}{X(x)} = -\lambda \quad (7) \quad \cancel{\text{and } \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}}$$

<u>and</u> $\dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)} = -\lambda$ (8), for some <u>constant</u> $\lambda \in \mathbb{R}$.

The boundary conditions (bc) on $u(x,t)$ imply:

$$0 = u(0,t) = X(0)\,T(t)$$
$$0 = u(L,t) = X(L)\,T(t) \quad , \text{ \underline{all} } t$$

$$\Longrightarrow \quad X(0) = X(L) = 0 \quad (\text{for non-trivial } T(t)).$$

Solving (4) is thus equivalent to solving two ordinary differential equations, one of which is a BVP:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} \qquad (9)$$

and ~~$T''(t) = \alpha^2 T(t)$~~

$$T'(t) + \alpha^2 \lambda T(t) = 0 \qquad (10)$$

We want to find a non-trivial sln. to (9) and (10) (i.e. a sln. which is not zero everywhere).

(A) Start with the sln. of (9):

linear 2nd order ODE (homog.):

charact. polynomial: $r^2 + \lambda = 0$

3 cases:

(1) $\lambda < 0 \Rightarrow r^2 = \underbrace{-\lambda}_{>0}$ - roots are real

$$r_{1,2} = \pm\sqrt{|\lambda|}$$

$\Rightarrow$ gen. sln. to (9): $X(x) = c_1 e^{\sqrt{|\lambda|}\,x} + c_2 e^{-\sqrt{|\lambda|}\,x}$ (11)

boundary conditions: $X(0) = X(L) \stackrel{!}{=} 0$

(11) only satisfies these BC. iff $c_1 = c_2 = 0$ (trivial sln.)

$\Rightarrow$ No non-trivial sln. for $\lambda < 0$.

② $\lambda = 0 \Rightarrow$ deq in (10) becomes

$$X''(x) = 0$$

$\Rightarrow$ general sln.: $X(x) = c_1 + c_2 x$ (12)

(12) satisfies the bc. iff $c_1 = c_2 = 0$.

$\Rightarrow$ No non-trivial sln. for $\lambda = 0$.

③ $\lambda > 0$: roots ~~for~~ of characteristic polynomial are complex, $r_{1,2} = \pm i\sqrt{\lambda}$.

$\Rightarrow$ general sln.:

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x) \quad (13)$$

boundary cond.:

$$0 = X(0) = c_1$$

$$0 = X(L) = \underbrace{c_1}_{=0} \cos(\sqrt{\lambda} \cdot L) + c_2 \sin(\sqrt{\lambda}\, L) = c_2 \sin(\sqrt{\lambda}\, L)$$

~~non-trivial X(x) given in (13)~~

The only non-trivial slns. to the BVP (9) ~~for~~ are obtained for $\lambda > 0$ and satisfy:

$$\sin(\sqrt{\lambda} \cdot L) = 0$$

$$\Leftrightarrow \quad \sqrt{\lambda} \cdot L = n\pi, \; n \in \mathbb{Z} \setminus \{0\}$$

want non-trivial sln.

$$\Leftrightarrow \quad \sqrt{\lambda} = \frac{n\pi}{L}, \quad n = \pm 1, \pm 2, \pm 3, \ldots$$

Therefore, the ~~fcns~~ family of fcns.

$$\left\{ \sin\left(\frac{n\pi x}{L}\right), \; n = 1, 2, 3, \ldots \right\}$$

forms a system of linearly independent solutions to the BVP in (9).

~~Now we learn~~

~~(B) Solve~~:

Note that the b.c. ~~imposed~~ restricts the permissible values for the parameter λ in ~~the heat eqn~~ (9) and hence in the heat eqn. (4).

Now we can solve

(B) eqn. (10): $T'(t) + \alpha^2 \lambda_n T(t) = 0$, where $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$

general sln. to (10): $T(t) = c\, e^{-\alpha^2 \lambda_n t} = c\, e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$

Let us summarize: The family of fcns.

$$\left\{ e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, 3, \ldots \right\}$$

forms a system of linearly independent slns. of (4). In particular, by above Prop. any finite linear combinations of these fcns.

$$\sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t},$$

some $N \in \mathbb{N}$ and $c_n \in \mathbb{R}$, is a sln. to (4).

Suppose, we are also given an initial condition for $u(x,t)$,

$$u(x,0) = f(x).$$

If ~~a~~ the sln. is of the form

$$u(x,t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

then $$f(x) = u(x,0) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) \quad (14)$$

We want the fcn. f to be piecewise cont., but not every piecewise cont. on $[0,L]$ will be a finite ~~le~~ linear comb. of $\sin\left(\frac{n\pi x}{L}\right)$ (in fact only very few of them will)

Could try to generalize (14) by writing

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \quad (15)$$

<u>questions:</u>

(i) Does this <u>formal expression</u> make sense (convergence ?, in what sense? ...)

(ii) Can every piecewise cont. fcn. on $[0,1]$ be written ~~as~~ in the form (15)?

This question was first raised by <u>Joseph Fourier</u> for exactly the ~~pu~~ purpose of solving the heat eqn.

Let us generalize (15) ~~to~~ to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\} \quad (16)$$

and ask questions (i) and (ii).

Let us suppose we ~~a~~ want to approximate f by a finite nb. of terms in (16)

$$~~\text{...}~~ \; s_n := \frac{a_0}{2} + \sum_{k=1}^{n} \left\{ a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right\}.$$

~~How could~~ We could quantify the quality of this approximation by calculating ~~a~~ the ~~mean sq~~ mean square deviation of s_n from f,

$$|f(x) - s_n(x)|^2, \; ~~\text{and}~~$$

averging over ~~o~~ the interval $[0, L)$, i.e.

$$\frac{1}{L}\int_0^L |f(x) - s_n(x)|^2 \, dx .$$ For this we require the fcn. f to be square integrable, i.e. $\int |f(x)|^2 \, dx < \infty$. We denote the class of fcns. with that property by $L^2([0,L])$. Every piecewise cont. fcn will be in ~~f~~ $L^2([0,L])$. We could then ~~understand~~ understand convergence of an approximation s_n to f as $n \to \infty$,

as ~~$s_n \to f$~~ $$\frac{1}{L}\int_0^L |f(x) - s_n(x)|^2 dx \xrightarrow{n\to\infty} 0$$

(mean square error goes to zero as $n \to \infty$)

It turns out that ~~this~~ ~~o~~ is the mathematically correct way to understand ~~the~~ convergence of the formal expression (16). One can prove the following to completely answer (i) and (ii):

<u>thm (Fourier-expansion):</u>

Let f be a square integrable fcn. on $[0,L]$ (i.e. $f \in L^2[0,L]$). Then, the sequence of approximate fcns.

$$S_n := \frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right\} \qquad (17)$$

converges to f in the sense

$$\int_0^L |S_n(x) - f(x)|^2 \, dx \longrightarrow 0 \quad \text{as } n \to \infty$$

if and only if the coefficients in (17) are given by

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx, \quad k = 0,1,2,\ldots$$

$$b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx, \quad k = 1,2,3,\ldots$$

We then write

$$(18) \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

and call this the <u>Fourier ~~expansion~~ series</u> for the fcn. f. The coefficients ~~in (18)~~ ~~are called~~ $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ are called <u>Fourier coefficients</u> of f.

Therefore, with this in mind we can write down the ~~general slu. sln. to general slan. to~~ solution to ~~the~~ BVP in (4) subject to the initial cond.

$u(x,0) = f(x)$, with piecewise cont. f on $[0,L]$ as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)$$

where $\cancel{f(x) = \sum_{n=1}}$ the coeff. (c_n) are the Fourier-coefficients of $\cancel{f}$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$