Introduction to differential equations

Higher order linear equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t), \quad a_i \in \mathbb{R}$$

– linear non-homogeneous differential equation of order n.

Homogeneous equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \qquad (h.e)$$

Characteristic polynomial:

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0 \qquad (c.p.)$$

If r is a root of (c.p.) of multiplicity m, then $e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}$ are solutions of (h.e)

If $r = \alpha + i\beta$ is a root of (c.p). of multiplicity m, then $r = (\alpha - i\beta)$ is also a root of (c.p.) of the same multiplicity, and

$e^{\alpha t}\cos\beta t,\ t e^{\alpha t}\cos\beta t, \dots, t^{m-1} e^{\alpha t}\cos\beta t,$

$e^{\alpha t}\sin\beta t,\ t e^{\alpha t}\sin\beta t, \dots, t^{m-1} e^{\alpha t}\cos\beta t$

are solutions of (h.e).

If $y_1, \dots, y_n$ are linearly independent solutions of (h.e) then the general solution is

$$y(t) = C_1 y_1 + C_2 y_2 + \dots + C_n y_n.$$

Non-homogeneous equations with constant coefficients and special right hand side.

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = e^{\alpha t} \cdot P(t) \sin \beta t$$

$$(\text{or } e^{\alpha t} \cdot P(t) \sin \beta t)$$

Let s be a multiplicity of $\alpha + i\beta$ as a root of characteristic polynomial ($s = 0$ if $\alpha + i\beta$ is not a root).

Then there is a partial solution of the form

$$y_p(t) = t^s e^{\alpha t}\left(T(t) \cos \beta t + R(t) \sin \beta t\right),$$

where $\deg T \le \deg P$ and $\deg R \le \deg P$.

Example

1) $y''' - 4y' = t$

$r^3 - 4r = 0$

$r(r-2)(r+2) = 0$

$\alpha + i\beta = 0$ – root of mult. 1, so

$y_p(t) = t(At + B)$

$y_p'(t) = 2At + B$

$y_p''(t) = 2A$

$$-4\cdot(2At+B)=t \quad , \quad A=-\frac{1}{8},\ B=0, \text{ so}$$

$$y_p(t)=-\frac{t^2}{8} \quad , \text{ and}$$

$$y(t)=C_1+C_2e^{2t}+C_3e^{-2t}-\frac{t^2}{8} \quad - \text{general solution.}$$

2) $y'''-4y'=3\cos t$

$r=i$ is not a root of char. polynomial, so

$$y_p(t)=A\cos t+B\sin t$$
$$y_p'(t)=-A\sin t+B\cos t$$
$$y_p''(t)=-A\cos t-B\sin t$$
$$y_p'''(t)=A\sin t-B\cos t$$

$$A\sin t-B\cos t-4(-A\sin t+B\cos t)=3\cos t$$

$$A=0, \quad B=-\frac{3}{5}, \text{ so} \quad y_p(t)=-\frac{3}{5}\sin t$$

3) $y'''-4y'=t+3\cos t$

$$y(t)=C_1+C_2e^{2t}+C_3e^{-2t}-\frac{t^2}{8}-\frac{3}{5}\sin t$$

– general solution.

Systems of differential equations

Problem

Find functions $x(t)$ and $y(t)$ such that

$$\begin{cases} \dfrac{dx}{dt} = ax + by + f(t) \\ \dfrac{dy}{dt} = cx + dy + g(t) \end{cases}$$

Method of elimination: reduce to a second-order linear differential equation

Examples

1) $$\begin{cases} \dfrac{dx}{dt} = x + y + t \\ \dfrac{dy}{dt} = x - y \end{cases}$$ Find all solutions.

$y' = x - y \quad \Rightarrow \quad x = y' + y$

$x' = x + y + t \quad \Rightarrow \quad y'' + y' = y' + y + y + t$

$y'' - 2y = t$

$r^2 - 2 = 0$

$y(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$ - general solution of homogeneous equation

Partial solution $y_p = At + B$,

$-2At - 2B = t$

$B = 0,\ A = -\frac{1}{2},\quad y_p = -\frac{t}{2}$

$$\begin{cases} y(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t} - \frac{t}{2} \\ x(t) = y' + y = (1+\sqrt{2})C_1 e^{\sqrt{2}t} + (1-\sqrt{2})C_2 e^{-\sqrt{2}t} - \frac{t}{2} - \frac{1}{2} \end{cases}$$

– general solution

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2) Find the solution of the initial-value problem

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = x \end{cases}, \quad x(0) = 1, \; y(0) = 1$$

<u>Solution:</u>

$$y'' = y' + y$$

$$y'' - y' - y = 0$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\begin{cases} y(t) = C_1 e^{\frac{1+\sqrt{5}}{2}t} + C_2 e^{\frac{1-\sqrt{5}}{2}t} \\ x(t) = C_1 \frac{1+\sqrt{5}}{2} e^{\frac{1+\sqrt{5}}{2}t} + C_2\left(\frac{1-\sqrt{5}}{2}\right) \cdot e^{\frac{1-\sqrt{5}}{2}t} \end{cases}$$

$$\begin{cases} C_1 + C_2 = 1 \quad \Rightarrow \quad C_1 = 1 - C_2 \\ C_1 \frac{1+\sqrt{5}}{2} + C_2 \frac{1-\sqrt{5}}{2} = 1 \end{cases}$$

$$\frac{1+\sqrt{5}}{2} - C_2\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

$$C_2\left(\frac{2\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2}$$

$$\begin{cases} C_2 = \dfrac{\sqrt{5}-1}{2\sqrt{5}} = \dfrac{5-\sqrt{5}}{10} \\ C_1 = 1 - C_2 = \dfrac{5+\sqrt{5}}{10}, \text{ so} \end{cases}$$

$$\begin{cases} y(t) = \cancel{\sqrt{5}-\sqrt{}}\ \dfrac{5+\sqrt{5}}{10} e^{\frac{1+\sqrt{5}}{2}t} + \dfrac{5-\sqrt{5}}{10} e^{\frac{1-\sqrt{5}}{2}t} \\ x(t) = \dfrac{\sqrt{5}+5}{10} \cdot \dfrac{1+\sqrt{5}}{2} e^{\frac{1+\sqrt{5}}{2}} + \dfrac{5-\sqrt{5}}{10} \cdot \dfrac{1-\sqrt{5}}{2} e^{\frac{1-\sqrt{5}}{2}t} = \\ \qquad = \dfrac{5+3\sqrt{5}}{10} e^{\frac{1+\sqrt{5}}{2}} + \dfrac{5-3\sqrt{5}}{10} e^{\frac{1-\sqrt{5}}{2}t} \end{cases}$$