Introduction to differential equations

Higher order linear equations

$$y^{(n)} + b_{n-1}(t)\, y^{(n-1)} + b_{n-2}(t)\, y^{(n-2)} + \dots + b_0(t)\, y = g(t)$$

– non-homogeneous linear equation of order n.

$n=1$

$$y' + a(t)y = g(t) \qquad (1)$$

Homogeneous equation: $y' + a(t)y = 0$,

$$y(t) = C \cdot e^{-\int a(t)dt}$$

Solutions of the homogeneous equation form a linear space.

If $y_1(t)$ is a solution of a non-homogeneous equation then $y(t) = C \cdot e^{-\int a(t)dt} + y_1(t)$ is a general solution of the non-homogeneous equation.

$n=2$

$$y'' + p(t)y' + q(t)y = g(t) \qquad (2)$$

Homog. equation

$$y'' + p(t)y' + q(t)y = 0$$

If $y_1(t), y_2(t)$ are solutions of the homog. equation then define the Wronskian

$$W[y_1, y_2] = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$W[y_1, y_2] \equiv 0$ or $W[y_1, y_2] \neq 0$ $\forall t$.

If $W[y_1, y_2] \neq 0$ then y_1, y_2 are linearly independent, and

$y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a general solution of the homogeneous equation.

If $y_p(t)$ is a solution of (2) then

$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$ is a general solution of (2).

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$$y^{(n)} + b_{n-1}(t) y^{(n-1)} + \ldots + b_0(t) y = g(t) \qquad (3)$$

If $y_1, \ldots, y_n$ are solutions then

$$W[y_1, \ldots, y_n] = \det \begin{pmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & & y_n^{(n-1)}(t) \end{pmatrix}$$

If $y_1, \ldots, y_n$ are linearly independent then

$W[y_1, \ldots, y_n](t) \neq 0$.

In fact, $W(t) = W(t_0)\, e^{-\int_{t_0}^{t} b_{n-1}(\tau) d\tau}$.

General solution

$y(t) = C_1 y_1 + \ldots + C_n y_n + y_{part}$

Homogeneous linear equations with constant coefficients.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 , \quad a_n \neq 0$$

We need to find n linearly independent solutions

Let us try to find a solution in the form
$y(t) = e^{rt}$.

$$e^{rt}(a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0) = 0$$

$$\underline{a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0}$$ characteristic polynomial

If $r_1, \dots, r_n$ are n distinct real roots then

$e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ are solutions, and

$$y(t) = C_1 e^{r_1 t} + \dots + C_n e^{r_n t}$$

is a general solution.

Example

$$y''' - 5y'' - 22y' + 56y = 0$$

Characteristic polynomial

$$r^3 - 5r^2 - 22r + 56 = 0$$

$(r+4)(r-2)(r-7) = 0$, so $-4, +2, +7$ – solutions, and

$$y(t) = C_1 e^{-4t} + C_2 e^{2t} + C_3 e^{7t}$$ is a general solution.

If r is a root of multiplicity k then

$$e^{rt}, te^{rt}, t^2e^{rt}, \dots, t^{k-1}e^{rt}$$

are solutions of the homogeneous equation

Example

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

Characteristic equation:

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r-1)(r+2)^3 = 0$$

Roots: $r_1 = \frac{1}{2}$, and $r_2 = -2$ of multiplicity 3.

General solution

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-2t} + C_3 t e^{-2t} + C_4 t^2 e^{-2t}$$

If $r = \alpha + i\beta$ is a root of characteristic polynomial of multiplicity k, then $\bar{r} = \alpha - i\beta$ is also a root of multiplicity k, and the functions

$$e^{\alpha t}\sin\beta t, \; te^{\alpha t}\sin\beta t, \dots, t^{k-1}e^{\alpha t}\sin\beta t$$

$$e^{\alpha t}\cos\beta t, \; te^{\alpha t}\cos\beta t, \dots, t^{k-1}e^{\alpha t}\cos\beta t$$

are linearly independent solutions.

Example

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = 0$$

$$r(r^2 + 6r + 34)^2 = 0$$

$$r_1 = 0, \quad \underbrace{r_{2,3} = -3 \pm 5i}$$ – multiplicity 2.

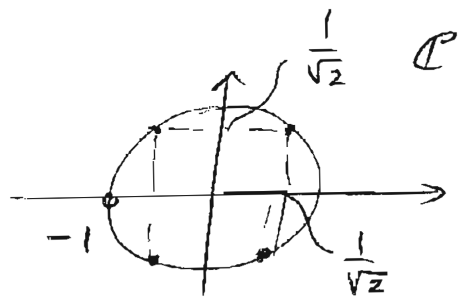
General solution:

$$y(t) = C_1 + C_2 e^{-3t} \sin 5t + C_3 e^{-3t} \cos 5t + C_4 t e^{-3t} \sin 5t + C_5 t e^{-3t} \cos 5t.$$

Example

$$y^{(4)} + y = 0$$

$$r^4 + 1 = 0$$



$$r_1 = \frac{1}{\sqrt{2}}(1+i),$$

$$r_2 = \frac{1}{\sqrt{2}}(1-i)$$

$$r_3 = \frac{1}{\sqrt{2}}(-1+i)$$

$$r_4 = \frac{1}{\sqrt{2}}(-1-i)$$

General solution:

$$y(t) = C_1 e^{\frac{t}{\sqrt{2}}} \sin \frac{t}{\sqrt{2}} + C_2 e^{\frac{t}{\sqrt{2}}} \cos \frac{t}{\sqrt{2}} + C_3 e^{-\frac{t}{\sqrt{2}}} \sin \frac{t}{\sqrt{2}} + C_4 e^{-\frac{t}{\sqrt{2}}} \cos \frac{t}{\sqrt{2}}.$$

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 =$$

$$= (r-1)(r-5)^2(r^2-4r+5) = 0$$

$r = 1$ – mult. 1

$r = 5$ – mult. 2

$r = 2 \pm i$

General solution:

$$y(t) = C_1 e^{t} + C_2 e^{5t} + C_3 t e^{5t} +$$
$$+ C_4 e^{2t} \cos t + C_5 e^{2t} \sin t.$$

Non-homogeneous equations with special right hand side.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 = e^{\alpha t} \cdot P(t) \cdot \sin \beta t$$

$$(\text{or } e^{\alpha t} \cdot P(t) \cos \beta t)$$

Let s be a multiplicity of $\alpha + i\beta$ as a root of characteristic polynomial ($s = 0$ if $\alpha + i\beta$ is not a root).

Then there is a partial solution of the form

$$y_p(t) = t^s e^{\alpha t}\left(T(t)\cos\beta t + R(t)\sin\beta t\right),$$

where $\deg T = \deg R = \deg P$.

Example

1) $y''' - 4y' = t$

$r^3 - 4r = 0$

$r(r-2)(r+2) = 0$

$\alpha + i\beta = 0$ – root of mult. 1, so

$y_p(t) = t(At + B)$

$y_p' = 2At + B$

$y_p'' = 2A$

$-4 \cdot 2At - 4B = t$, $A = -\frac{1}{8}$, $B = 0$, so

$$y_p(t) = -\frac{t^2}{8}, \text{ and}$$

$$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t} - \frac{t^2}{8}$$

is a general solution

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2) $y''' - 4y' = 3\cos t$

i is not a root, so

$y_p(t) = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$y_p''' = A\sin t - B\cos t$

$A\sin t - B\cos t - 4(-A\sin t + B\cos t) = 3\cos t$

$A = 0, \quad B = -\frac{3}{5}$, so

$y_p(t) = -\frac{3}{5}\sin t,$

3) $y''' - 4y' = t + 3\cos t$

$y_p(t) = -\frac{t^2}{8} - \frac{3}{5}\sin t,$

$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t} - \frac{t^2}{8} - \frac{3}{5}\sin t$

– general solution