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Introduction to differential equations

Laplace transform

Def $f(t)$, $t \geq 0$ - piecewise continuous, of exponential order

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Properties of Laplace transform

1) $\mathcal{L}(1) = \frac{1}{s}$, $s > 0$

2) $\mathcal{L}(e^{ta}) = \frac{1}{s-a}$, $s > a$

3) $\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$, $\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$, $s > 0$

4) $\mathcal{L}(a_1 f_1(t) + a_2 f_2(t)) = a_1 \mathcal{L}(f_1(t)) + a_2 \mathcal{L}(f_2(t))$

5) $F(s) = \mathcal{L}(f(t)) \Rightarrow$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$$

6) $F(s) = \mathcal{L}(f(t)) \Rightarrow$

$$\mathcal{L}(-t f(t)) = \frac{d}{ds} F(s)$$

7) $a y'' + b y' + c y = f(t)$, $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}(y(t)) = Y(s)$
$y(0) = y_0$, $y'(0) = y_0'$

$$Y(s) = \frac{(as+b)y_0 + a y_0' + F(s)}{as^2 + bs + c}$$

8) $\mathcal{L}(f(t)) = F(s) \Rightarrow$

$$\mathcal{L}(e^{at} f(t)) = F(s-a)$$

9) $\mathcal{L}(f(t)) = F(s) \Rightarrow$

$$\mathcal{L}(H_c(t) \cdot f(t-c)) = e^{-cs} F(s),$$

where $H_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$ – Heaviside function.

Examples

1)

$$y' - y = \cos t, \quad y(0) = 0$$

Denote $Y(s) = \mathcal{L}(y(t))$.

$$\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(\cos t)$$

$$Y(s) \cdot s - \underbrace{y(0)}_{=0} - Y(s) = \frac{s}{s^2+1}$$

$$Y(s)(s-1) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} =$$

$$= \frac{A(s^2+1) + (s-1)(Bs+C)}{(s-1)(s^2+1)}$$

We have $\begin{cases} A + B = 0 & \Rightarrow B = -A \\ -B + C = 1 & \Rightarrow 2A = 1,\ A = \frac{1}{2}, \text{ so} \\ A - C = 0 & \Rightarrow A = C \end{cases}$

$\begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \\ C = \frac{1}{2} \end{cases}$, so $Y(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{s-1}{s^2+1}\right)$

$$Y(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) =$$

$$= \mathcal{L}\left(\frac{1}{2}\left(e^t - \cos t + \sin t\right)\right), \text{ so}$$

$$y(t) = \frac{1}{2}\left(e^t - \cos t + \sin t\right)$$

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2) $$y'' + 4y = f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & t \ge 4 \end{cases},$$

$$y(0) = 3, \; y'(0) = -2$$

$$f(t) = H_0(t) - H_4(t), \text{ so}$$

$$\mathcal{L}(f(t)) = \mathcal{L}(H_0(t) - H_4(t)) =$$

$$\mathcal{L}(H_c(t)) = e^{-sc} \cdot \frac{1}{s}, \text{ so}$$

$$\mathcal{L}(f(t)) = \frac{1}{s} - e^{-4s} \cdot \frac{1}{s} = \frac{1-e^{-4s}}{s}.$$

$$s^2 Y(s) - s\, y(0) - y'(0) + 4Y(s) = \frac{1-e^{-4s}}{s}$$

$$(s^2+4)\, Y(s) = \frac{1-e^{-4s}}{s} + 3s - 2$$

$$Y(s) = \frac{1}{s^2+4}\left(\frac{1}{s} + 3s - 2\right) - e^{-4s}\frac{1}{s(s^2+4)} =$$

$$= 3\frac{s}{s^2+4} - \frac{2}{s^2+4} + (1-e^{-4s})\left(\frac{s^2+4-s^2}{4s(s^2+4)}\right) =$$

$$= 3\frac{s}{s^2+4} - \frac{2}{s^2+4} + (1-e^{-4s})\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) =$$

$$= \mathcal{L}\Big(\underbrace{3\cos 2t - 2\sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - H_4(t)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-4)\right)}_{u(t)}\Big)$$

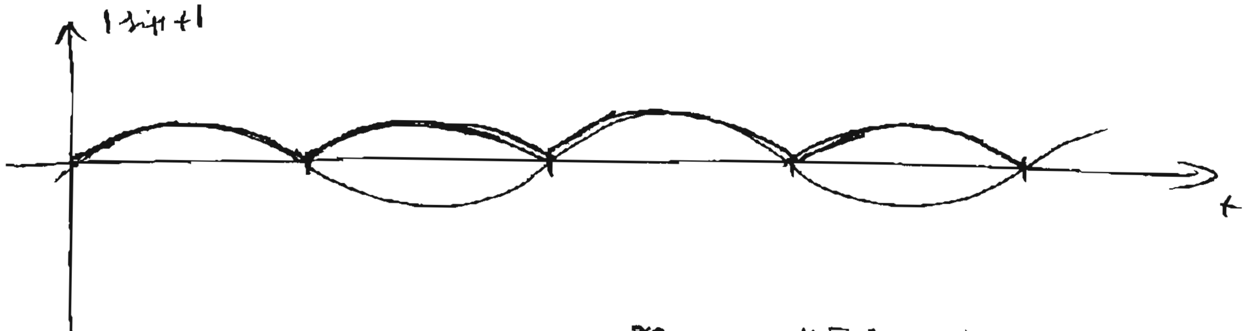
Remark

Both $y(t)$ and $y'(t)$ are continuous for $t \geq 0$!

3) $f(t) = |\sin t|$, find $\mathcal{L}(f(t))$.

Solution:

$$f(t) = |\sin t| = \sin t + 2 \sum_{n=1}^{\infty} H_{n\pi}(t) \cancel{\sin}(t - n\pi)$$



$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} + 2 \sum_{n=1}^{\infty} e^{-n\pi s} \cdot \frac{1}{1+s^2} =$$

$$= \frac{1}{1+s^2}\left(1 + 2 \sum_{n=1}^{\infty} e^{-n\pi s}\right) =$$

$$= \frac{1}{1+s^2}\left(1 + 2\,\frac{e^{-\pi s}}{1 - e^{-\pi s}}\right) = \frac{1}{1+s^2} \cdot \frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

4) $y'' + 2y' + y = 2(t-3)\,H_3(t)$, $y(0) = 2$, $y'(0) = 1$

$$s^2 Y(s) - s\,y(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) =$$
$$= 2\,e^{-3s} \cdot \frac{1}{s^2}$$

$$(s^2 + 2s + 1)\,Y(s) = \frac{2e^{-3s}}{s^2} + 2s + 1 + 4 =$$
$$= \frac{2e^{-3s}}{s^2} + 2s + 5$$

$$Y(s) = \frac{2e^{-3s}}{s^2(s+1)^2} + \frac{2s+5}{(s+1)^2} =$$

$$= \frac{2e^{-3s}((s+1)^{\cancel{2}} - s^{\cancel{2}})}{s^2(s+1)^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2} =$$

$$= 2e^{-3s}\left(\frac{1}{s^2(s+1)} - \frac{1}{s(s+1)^2}\right) + \frac{2}{s+1} + \frac{3}{(s+1)^2} =$$

$$= 2e^{-3s}\left(\frac{1}{s^2} - \frac{2}{s(s+1)} + \frac{1}{(s+1)^2}\right) + \frac{2}{(s+1)} + \frac{3}{(s+1)^2} =$$

$$= 2e^{-3s}\left(\frac{1}{s^2} + \frac{1}{(s+1)^2} - 2\left(\frac{1}{s} - \frac{1}{s+1}\right)\right) + \frac{2}{s+1} + \frac{3}{(s+1)^2} =$$

$$= \mathcal{L}\left(2H_3(t)\left((t-3) + e^{-(t-3)}(t-3) - 2 + 2e^{-(t-3)}\right) + 2e^{-t} + 3e^{-t}t\right),$$

$$y(t) = 2H_3(t)\left((t-5) + (t-1)\,e^{3-t}\right) + e^{-t}(2+3t)$$

5) $y''' - y = 0 \quad , \quad y(0) = 1, \; y'(0) = 0, \; y''(0) = 0$ 6

$\mathcal{L}(f') = sF(s) - f(0)$

$\mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}(f''') = \mathcal{L}((f')'') = s^2\mathcal{L}(f') - sf'(0) - f''(0) =$

$= s^2(sF(s) - f(0)) - sf'(0) - f''(0) =$

$= s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

$\mathcal{L}(y''') - \mathcal{L}(y) = 0$

$s^3Y(s) - s^2y(0) - \cancel{s\,y'(0)}^{\,0} - \cancel{y''(0)}^{\,0} - Y(s) = 0$

$(s^3 - 1)Y(s) = s^2$

$Y(s) = \dfrac{s^2}{s^3-1} = \dfrac{s^2}{(s-1)(s^2+s+1)} =$

$= \cancel{(s^2+s+1) \quad (s+1)} \; \dfrac{A}{s-1} + \dfrac{Bs+C}{(s^2+s+1)} =$

$= \dfrac{A(s^2+s+1) + (s-1)(Bs+C)}{(s-1)(s^2+s+1)}$

$$\begin{cases} A + B = 1 & A + 2A = 1, \; A = \frac{1}{3} \\ A + C - B = 0 & \Rightarrow B = A + C = 2A, \quad B = \frac{2}{3} \\ A - C = 0 & \Rightarrow A = C \qquad C = \frac{1}{3} \end{cases}$$

$Y(s) = \dfrac{1}{3}\left(\dfrac{1}{s-1} + \dfrac{2s+1}{s^2+s+1}\right) =$

$= \dfrac{1}{3}\left(\dfrac{1}{s-1} + \dfrac{2(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{3}{4}}\right) = \mathcal{L}\left(\dfrac{e^t}{3} + \dfrac{2}{3}e^{-\frac{t}{2}} \times \cos\dfrac{\sqrt{3}}{2}t\right)$

$y(t) = \dfrac{e^t}{3} + \dfrac{2}{3}e^{-t/2}\cos\dfrac{\sqrt{3}}{2}t$