Introduction to differential equations

Dec. 5, 08

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Final exam review

2.1

$$\cos y \sin t \frac{dy}{dt} = \sin y \cos t$$

$$\frac{\cos y}{\sin y}\frac{dy}{dt} = \frac{\cos t}{\sin t}$$

$$\int \frac{\cos y \, dy}{\sin y} = \int \frac{\cos t \, dt}{\sin t} + C$$

$$\ln|\sin y| = \ln|\sin t| + C$$

$$\boxed{\sin y = C \sin t}$$

2.2

$$y_1 = t e^t, \quad y_2 = (t-2)e^t$$

$$t y'' - (t+1)y' + y = (t-1)e^t$$

Since y_1, y_2 are solutions of the non-homogeneous equation, $y_1 - y_2$ is a solution of the homogeneous equation

$$t y'' - (t+1)y' + y = 0,$$

$$y_1 - y_2 = 2e^t$$

Let us take $z(t) = y(t)e^{-t}$,

$$y(t) = e^{t} z(t),$$

$$y' = e^{t} z + e^{t} z'$$

$$y'' = e^{t} z + 2e^{t} z' + e^{t} z''$$

$$t(e^{t} z + 2e^{t} z' + e^{t} z'') - (t+1)(e^{t} z + e^{t} z') + e^{t} z = 0$$

$$t(z + 2z' + z'') - (t+1)(z + z') + z = 0$$

$$tz'' + \underline{2tz'} + \cancel{tz} - \cancel{tz} - \underline{tz'} - \cancel{z} - z' + \cancel{z} = 0$$

$$tz'' + tz' - z' = 0$$

$$tz'' + (t-1)z' = 0$$

$$u(t) = z'(t)$$

$$tu' + (t-1)u = 0$$

$$\int \frac{du}{u} = \int \frac{1-t}{t}\,dt + C$$

$$\ln|u| = \ln|t| - t + C$$

$$u(t) = C t e^{-t}$$

$$z(t) = \int u(t)\,dt + C_2$$

$$z(t) = C_1 \int t e^{-t}\,dt + C_2 =$$

$$= -C_1(te^{-t} + e^{-t}) + C_2$$

$$\underline{y(t) = e^{t} z(t) = C_1(t+1) + C_2 e^{t}}$$

$$\int t e^{-t}\,dt =$$

$$= -\int t\, de^{-t} =$$

$$= -\left(te^{-t} - \int e^{-t}\,dt\right)$$

$$= -\left(te^{-t} + e^{-t}\right) =$$

$$= -te^{-t} - e^{-t}$$

So the general solution of the initial equation is

$$y(t) = C_1(t+1) + C_2 e^{t} + te^{t}$$

2.4.

$$t^2 y'' - 4ty' + 6y = 0$$

$$y(t) = t^2$$

$$y(t) = t^2 z(t)$$

$$y' = 2tz + t^2 z'$$

$$y'' = 2z + 4tz' + t^2 z''$$

$$t^2(2z + 4tz' + t^2 z'') - 4t(2tz + t^2 z') + 6t^2 z = 0$$

$$\cancel{2z} + \underline{4tz'} + t^2 z'' - \cancel{8z} - \underline{4tz'} + \cancel{6z} = 0$$

$$t^2 z'' = 0$$

$$z'' = 0$$

$$z(t) = C_1 t + C_2$$

$$\underline{y(t) = (C_1 t + C_2) \cdot t^2}$$

2.5

$$\dot{\bar{x}} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x}, \quad \bar{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{pmatrix} =$$

$$= \lambda(\lambda - 1) - 2 = \lambda^2 - \lambda - 2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \{-1, 2\},$$

Eigenvalue for $\lambda = -1$

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ take } \bar{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Eigenvalue for $\lambda = 2$

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$ take $\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution

$$\bar{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $\bar{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ then

$$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 + c_2 = -1 \end{cases} \Rightarrow \underline{c_1 = 1 = c_2},$$

so $$\bar{x}(t) = \begin{pmatrix} e^{-t} + e^{2t} \\ -2e^{-t} + e^{2t} \end{pmatrix}$$