Introduction to differential equations

Final exam review.

- First order differential equations
  - Separable equations
  - Linear homogeneous and non-homogeneous equations
  - Exact equations
  - Homogeneous equations
- The existence and uniqueness Theorem for the first order differential equations; Picard iterations.
- Second order differential equations, properties of solutions, linear independence, Wronskian; method of reduction of order.
- Second order differential equations with constant coefficients, characteristic polynomial, general solution of a homogeneous equation.
- Non-homogeneous linear second order differential equations with constant coefficients; method of variation of parameters.

- Non-homogeneous linear —"—, special r.h.s. (= "method of judicious guessing").

- Series solutions

- The method of Laplace transform: Laplace transform of some elementary functions, properties of Laplace transform, Laplace transform of the Heaviside function, application to solving the IVP for linear differential equations.

- Higher-order equations

- Systems of linear differential equations with constant coefficients.

---

5 problems (50 points max)

2 hours

no textbooks etc.

1.1.

$$t y'' + 2y' = 0$$

$$z(t) = y'(t)$$

$$t z' + 2z = 0$$

$$\frac{z'}{z} = -\frac{2}{t}$$

$$\int \frac{dz}{z} = -\int \frac{2dt}{t} + C$$

$$\ln|z| = -\ln t^2 + C$$

$$|z| = e^{C} \cdot \frac{1}{t^2}$$

$$z = \tilde{C} \cdot \frac{1}{t^2}$$

$$y' = \frac{\tilde{C}}{t^2}, \quad \boxed{y(t) = -\frac{\tilde{C}}{t} + \tilde{\tilde{C}} = C_1 + \frac{C_2}{t}}$$

1.2.

$$\begin{aligned} y' &= t - y^2 \\ y(0) &= 0 \end{aligned}$$

$$L(t, y(t)) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$$

$$y_0(t) \equiv 0$$

$$y_1(t) = y(0) + \int_0^t (s - y_0^2(s))\,ds = \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

$$y_2(t) = y(0) + \int_0^t (s - y_1^2(s))\,ds = \int_0^t \left(s - \frac{s^4}{4}\right)ds =$$

$$= \left(\frac{s^2}{2} - \frac{s^5}{20}\right)\Big|_0^t = \frac{t^2}{2} - \frac{t^5}{20}$$

1.3.

$$F(s) = \frac{s}{s^2+4s+8}, \quad \mathcal{L}^{-1}(F(s)) = ?$$

$$F(s) = \frac{s}{s^2+4s+8} = \frac{s}{(s+2)^2+4} = \frac{s+2}{(s+2)^2+2^2} - \frac{2}{(s+2)^2+2^2} =$$

$$= \mathcal{L}\left(\underline{e^{-2t}\cos 2t - e^{-2t}\sin 2t}\right)$$

Properties of Laplace transform

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

If $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(-t f(t)) = \frac{d}{ds} F(s)$

If $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(e^{at} f(t)) = F(s-a)$

$$H_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}, \quad \mathcal{L}(H_c(t)) = \frac{e^{-cs}}{s}$$

$$\mathcal{L}(H_c(t) f(t-c)) = e^{-cs} F(s), \text{ if } F(s) = \mathcal{L}(f(t))$$

$$\mathcal{L}(f'(t)) = s\,\mathcal{L}(f(t)) - f(0) = s F(s) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$$

1.4.

$$y'' - 2t y' + \lambda y = 0$$

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots = \sum_{k=0}^{\infty} a_k t^k$$

$$y'(t) = \sum_{k=1}^{\infty} a_k \cdot k t^{k-1}$$

$$y''(t) = \sum_{k=2}^{\infty} a_k (k-1) k \, t^{k-2}$$

$$y'' - 2t y' + \lambda y = 0 =$$

$$= \sum_{k=2}^{\infty} a_k (k-1) k t^{k-2} - 2t \sum_{k=1}^{\infty} a_k k t^{k-1} + \lambda \sum_{k=0}^{\infty} a_k t^k =$$

$$= \sum_{k=0}^{\infty} a_{k+2} (k+1)(k+2) t^k - \cancel{2 \sum_{k=1}^{\infty} a_k (k+1) t^k}$$

$$- 2 \sum_{k=0}^{\infty} a_k \cdot k \cdot t^k + \lambda \sum_{k=0}^{\infty} a_k t^k =$$

$$= \sum_{k=0}^{\infty} \left(a_{k+2} (k+1)(k+2) - 2 a_k \cdot k + \lambda a_k \right) t^k$$

$$\boxed{a_{k+2} = - \frac{\lambda - 2k}{(k+1)(k+2)} a_k}$$

Take $y(t)$ s.t. $y(0) = 1$, $y'(0) = 0$, i.e. $a_0 = 1$, $a_1 = 0$.

Then $a_{2n+1} = 0 \ \forall n$, and

$$y(t) = 1 - \frac{\lambda}{2} t^2 + \frac{\lambda(\lambda - 4)}{4!} t^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6!} t^6 + \ldots$$

Take $y(t)$ s.t. $y(0)=0,\ y'(0)=1$, (6)

then $a_{2n}=0\ \forall n$,

$$y(t) = t - \frac{\lambda-2}{2\cdot 3}t^3 + \frac{(\lambda-2)(\lambda-6)}{5!}t^5 - \cdots$$

$$- \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7!}t^7 + \cdots$$

1.5.

$$\frac{d}{dt}\bar{x} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\bar{x}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{pmatrix} =$$

$$= (1-\lambda)^2 + 9 = \lambda^2 - 2\lambda + 10,$$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$$

Take $\lambda = 1+3i$

$$\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3i v_1 - 3v_2 = 0 \\ 3v_1 - 3i v_2 = 0 \end{cases}, \text{ take } \bar{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Therefore $e^{\lambda t}\bar{v}$ is a solution,

$$e^{\lambda t}\bar{v} = e^{(1+3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^t(\cos 3t + i\sin 3t)\begin{pmatrix} i \\ 1 \end{pmatrix} =$$

$$= e^{t}\begin{pmatrix} i\cos 3t - \sin 3t \\ \cos 3t + i\sin 3t \end{pmatrix} =$$

$$= e^{t}\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + i e^{t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix},$$

so the general solution is

$$\overline{x}(t) = c_1 e^{t}\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + c_2 e^{t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} =$$

$$= \begin{pmatrix} -c_1 e^{t}\sin 3t + c_2 e^{t}\cos 3t \\ c_1 e^{t}\cos 3t + c_2 e^{t}\sin 3t \end{pmatrix}$$

2.3.

$$F(s) = \frac{4}{(s-1)^2}$$

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad \mathcal{L}(e^t \cdot t) = \frac{1}{(s-1)^2},$$

$$\mathcal{L}(4e^t \cdot t) = \frac{4}{(s-1)^2}$$

---

3. 3

$$F(s) = \frac{s}{(s-1)^2} = \frac{s-1+1}{(s-1)^2} = \frac{1}{s-1} + \frac{1}{(s-1)^2} =$$

$$= \mathcal{L}(e^t + te^t)$$

---

3. 2

$$W[y_1, y_2] = \det\begin{pmatrix} e^t \cos t & e^t \sin t \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t \end{pmatrix} =$$

$$= e^{2t}(\cos t \sin t + \cos^2 t - \cos t \sin t + \sin^2 t) = e^{2t}$$
