Linear systems of differential equations.

$$(*) \quad \frac{d}{dt}\bar{x} = A\bar{x}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

If λ is an eigenvalue of A that corresponds to the eigenvector $\bar{v}$ then

$\bar{x}(t) = e^{\lambda t}\bar{v}$ is a solution.

If $\lambda_1, \dots, \lambda_n$ are different eigenvalues corresponding to eigenvectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$

then $\bar{x}(t) = C_1 e^{\lambda_1 t}\bar{v}_1 + C_2 e^{\lambda_2 t}\bar{v}_2 + \dots + C_n e^{\lambda_n t}\bar{v}_n$

is a general solution.

If $\lambda = \alpha + i\beta$ is an eigenvalue, and $\bar{v} = \bar{v}_1 + i\bar{v}_2$ is a complex eigenvector, then

$e^{\lambda t}\bar{v}$ is a complex-valued solution, and

$Re(e^{\lambda t}\bar{v})$ and $Im(e^{\lambda t}\bar{v})$ are

linearly independent real-valued solutions of (*

Assume that λ is a solution of the equation

$$\det(A - \lambda I) = 0$$

of multiplicity $k > 1$.

(2)

If $\exists\, k$ eigenvectors that corresponds to λ which are linearly independent, $\bar{v}_1, \dots, \bar{v}_k$ then each of the functions

$$e^{\lambda t}\bar{v}_1, \dots, e^{\lambda t}\bar{v}_k \quad \text{is a solution.}$$

<u>Example</u>

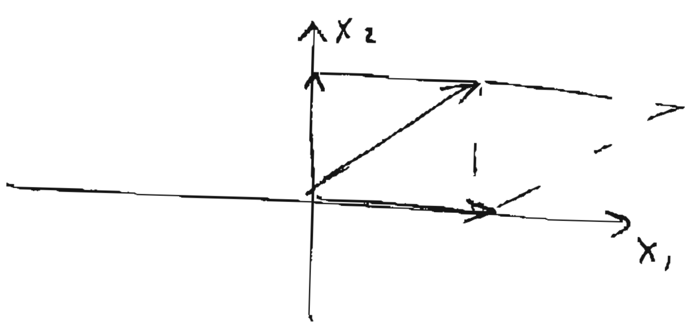
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 = 0,$$

$$\underline{\lambda_{1,2} = 1}.$$

There is only one (up to multiplication by a constant) eigenvector:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



If $\bar{v}_1$ is an eigenvector, then

$$e^{\lambda t}\bar{v}_1 \text{ is a solution.}$$

Let us try to find linearly independent solution in a form

$$\bar{x}(t) = e^{\lambda t}(t\bar{v}_1 + \bar{v}_2)$$

$$\frac{d}{dt}\bar{x}(t) = \underline{\lambda e^{\lambda t} t \bar{v}_1} + e^{\lambda t}\bar{v}_1 + \lambda e^{\lambda t}\bar{v}_2 =$$

$$= A\bar{x}(t) = A\left(e^{\lambda t}(t\bar{v}_1 + \bar{v}_2)\right) =$$

$$= \underline{e^{\lambda t}\cdot t \cdot \lambda \bar{v}_1} + e^{\lambda t} A\bar{v}_2,$$

so $\bar{v}_1 + \lambda\bar{v}_2 = A\bar{v}_2$

$$\boxed{(A - \lambda I)\bar{v}_2 = \bar{v}_1}$$

Therefore, if $\bar{v}_1$ is an eigen~~value~~vector that corresponds to λ, and

$\bar{v}_2$ is a solution of ,

then $e^{\lambda t}(t\bar{v}_1 + \bar{v}_2)$ is another solution.

If we considered all linearly independent solutions of $(A - \lambda I)\bar{v}_2 = \bar{v}_1$, ~~the~~ and did not get k linearly independent solutions then we can try to find a solution in a form

$$e^{\lambda t}(\bar{v}_1 t^2 + \bar{v}_2 t + \bar{v}_3), \text{ and so on.}$$

Remark

Another description for $\bar{v}_2$ is the following

$$(A - \lambda I)^2 \bar{v}_2 = \bar{0}$$

If this does not give enough solutions, we can consider

$$(A - \lambda I)^3 \bar{v}_3 = \bar{0}, \text{ and so on.}$$

Example

$$\frac{d}{dt}\bar{x} = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\bar{x}$$

$$\det(A - \lambda I) = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2, \quad \lambda_{1,2} = 5$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ take } \bar{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Let us find $\bar{w}$ such that $(A - \lambda I)\bar{w} = \bar{v}$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} 2w_1 + w_2 = 1 \\ -4w_1 - 2w_2 = -2 \end{cases}, \quad \bar{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left(\text{or } \bar{w} = \begin{pmatrix} w_1 \\ 1 - 2w_1 \end{pmatrix}\right)$$

We have solutions:

$$e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} \quad \text{and} \quad e^{5t}\left(t\begin{pmatrix}1\\-2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right),$$

so the general solution is

$$\bar{x}(t) = C_1 e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + C_2\left(e^{5t}t\begin{pmatrix}1\\-2\end{pmatrix} + e^{5t}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

Example

$$\frac{d}{dt}\bar{x} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}\bar{x}$$

$$\det(A-\lambda I) = 0, \quad \lambda^2 - 4\lambda + 4 = 0$$

$$\underline{\lambda_{1,2} = 2}$$

Eigenvector:

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \bar{0}, \quad \text{take } \bar{v} = \begin{pmatrix}1\\0\end{pmatrix}.$$

Let us find $\bar{w} = \begin{pmatrix}w_1\\w_2\end{pmatrix}$ such that

$$(A-\lambda I)\bar{w} = \bar{v}$$

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$w_2 = 1$ take $\bar{w} = \begin{pmatrix}0\\1\end{pmatrix}$

General solution:

$$\bar{x}(t) = C_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + C_2\left(e^{2t}\cdot t\begin{pmatrix}1\\0\end{pmatrix} + e^{2t}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

On exponent of a matrix

$\frac{d}{dt}x = ax \;,\; x(0) = x_0$

$\underline{x(t) = e^{at} \cdot x_0}$ – solution

$\frac{d}{dt}\bar{x} = A\bar{x} \;,\; \bar{x}(0) = \bar{x}_0$

Solution: $\underline{\bar{x}(t) = e^{At} \cdot \bar{x}_0}$

How to define e^{At}?

<u>Def</u>

$$e^{A} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

<u>Example</u>:

1) $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$, $e^{A} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \; e^{\lambda_n} \end{pmatrix}$

2) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\cancel{A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $e^{A} = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

3) $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$$e^{A} = e^{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} \cdot e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = e^{\lambda} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix}$$

<u>Remark</u>: If $AB = BA$ then $e^{A+B} = e^{A} \cdot e^{B} = e^{B} \cdot e^{A}$.

Example

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

$$\lambda^2 - 6\lambda + 9 = 0, \quad (\lambda - 3)^2 = 0$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

take $\bar{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

let us find $\bar{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ such that

$$(A - \lambda I)\bar{w} = \bar{v}$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

take $\bar{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

general solution is

$$\bar{x}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(e^{3t} t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$