ANALYTIC FUNCTIONS (MATH 220 B)

Final exam (sample)

In all the problems D is the unit disc, $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

Problem 1.

Let $L \subset \mathbb{C}$ be the line $L = \{z = x + iy \mid x = 3\}$. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that for any $z \in L$ we have $f(z) \in L$. Assume that $f(0) = 1 + 2i$. Find $f(6)$.

Problem 2.

Suppose a function $f : \overline{D} \to \mathbb{C}$ is continuous and holomorphic in D. Suppose also that for any $z \in \partial D$ we have $\mathbf{Re} f(z) = 5$. Prove that f is a constant.

Problem 3.

Let $f : \{z = x + iy \mid y > 0\} \to \mathbb{C}$ be a holomorphic function such that $f\left(\frac{i}{\sqrt{n}}\right) = 0$ for every $n \in \mathbb{N}$. Prove that f is unbounded.

Problem 4.

a) TRUE OR FALSE: A continuous function $v : \mathbb{R} \to \mathbb{R}$ is linear if and only if for every $x, h \in \mathbb{R}$ one has

$$\frac{1}{2}\left(v(x-h) + v(x+h)\right) = v(x).$$

b) TRUE OR FALSE: A continuous function $u : \mathbb{C} \to \mathbb{R}$ is harmonic if and only if for every $z, h \in \mathbb{C}$ one has:

$$\frac{1}{4}\left(u(x-h) + u(x+h) + u(x-ih) + u(x+ih)\right) = u(x).$$

Problem 5.

Let $\{f_\alpha\}_{\alpha \in A}$ be a family of holomorphic functions on D such that

$$\forall x \in D \;\; \forall f \in \{f_\alpha\} \quad \mathbf{Re} f(z) \neq \mathbf{Im} f(z).$$

Prove that $\{f_\alpha\}_{\alpha \in A}$ is a normal family (use the definition of normal family that allows a sequence of function to converge to ∞, uniformly on compact sets).