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Research statement

Here I mention only results that were published or obtained during the period July 1, 2007 – September 30, 2010. A brief description of results completed before 2007 can be found on my webpage (*www.math.uci.edu/∼asgor*).

Spectral properties of the discrete Schrödinger operator with Fibonacci potential

It is always exciting to obtain a new connection between two different branches of mathematics. Here I describe a series of new results concerning the spectral properties of the discrete Schrödinger operator with Fibonacci potential, the so-called Fibonacci Hamiltonian, obtained by methods of the modern theory of dynamical systems (uniformly hyperbolic and normally hyperbolic dynamics).

The Fibonacci Hamiltonian is a central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded self-adjoint operator in $\ell^2(\mathbb{Z})$,

$$[H_{V,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + V\chi_{[1-\alpha,1)}(n\alpha + \omega \mod 1)\psi(n),$$

where $V > 0$, $\alpha = \frac{\sqrt{5}-1}{2}$, and $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. The spectrum is easily seen to be independent of ω and may therefore be denoted by Σ_V. It is known that Σ_V is a zero-measure Cantor set for every $V \neq 0$.

Naturally, one is interested in quantitative fractal properties of Σ_V, such as its dimension, thickness, and denseness. While such a study is well-motivated from a purely mathematical perspective, there is also significant additional interest in these quantities. In particular, in a paper [1] joint with Damanik, Embree and Tcheremchantsev we show that *the fractal dimension of the spectrum is intimately related with the long-time asymptotics of the solution to the associated time-dependent Schrödinger equation, that is,* $i\partial_t\phi = H_{V,\omega}\phi$.

Results by Casdagli and Sütő show that for $V \geq 16$, Σ_V is a dynamically defined Cantor set. This implies that the Hausdorff dimension and the upper and lower box counting dimension of Σ_V all coincide; let us denote this common value by $\dim \Sigma_V$. Using this result, in [1] we have shown upper and lower bounds for the dimension. A particular consequence of these bounds is the identification of the asymptotic behavior of the dimension as V tends to infinity:

$$\lim_{V\to\infty} \dim \Sigma_V \cdot \log V = \log(1+\sqrt{2}).$$

This represents the final word in a topic with a long and reach history. One of the ingredients of the proof is application of results well known in hyperbolic dynamics to the trace map of the Fibonacci Hamiltonian.

In a joint paper with Damanik [2] we also prove that *the spectrum* Σ_V *of the Fibonacci Hamiltonian* H_V *is a dynamically defined Cantor set for small enough values of the coupling constant*, and obtain some new properties of the spectrum (in particular, smooth dependence of the Hausdorff dimension of the spectrum on coupling constant) for these values of V. In [3] we study the spectrum and the spectral type of the off-diagonal Fibonacci operator and obtain similar results[1].

[1] The content of [3] is published as an appendix to [5], not as a separate paper.

Clearly, as V approaches zero, $H_{V,\omega}$ approaches the free Schrödinger operator

$$[H_0\psi](n) = \psi(n+1) + \psi(n-1),$$

which is a well-studied object whose spectral properties are completely understood. In particular, the spectrum of H_0 is given by the interval $[-2, 2]$. Lebesgue measure of the spectrum does not extend continuously to the case $V = 0$. Given this situation, one would at least hope that the dimension of the spectrum is continuous at $V = 0$. In [4, 5] we prove that *the thickness tends to infinity and, hence, the Hausdorff dimension of the spectrum tends to one,* $\lim_{V\to 0} \dim_H \Sigma_V = 1$.

Also, it is natural to ask about the size of the gaps in the spectrum Σ_V, which can in fact be parametrized by a canonical countable set of gap labels. These gap openings were studied by Bellissard for a Thue-Morse potential and by Bellissard-Bovier-Ghez for period doubling potential. However, for the important Fibonacci case, the problem remained open. In fact, Bovier and Ghez remarked: *It is a quite perplexing feature that even in the simplest case of all, the golden Fibonacci sequence, the opening of the gaps at small coupling is not known!* In [5] we prove that *at small coupling, all gaps allowed by the gap labeling theorem are open and the length of every gap tends to zero linearly*, see Figure 1.

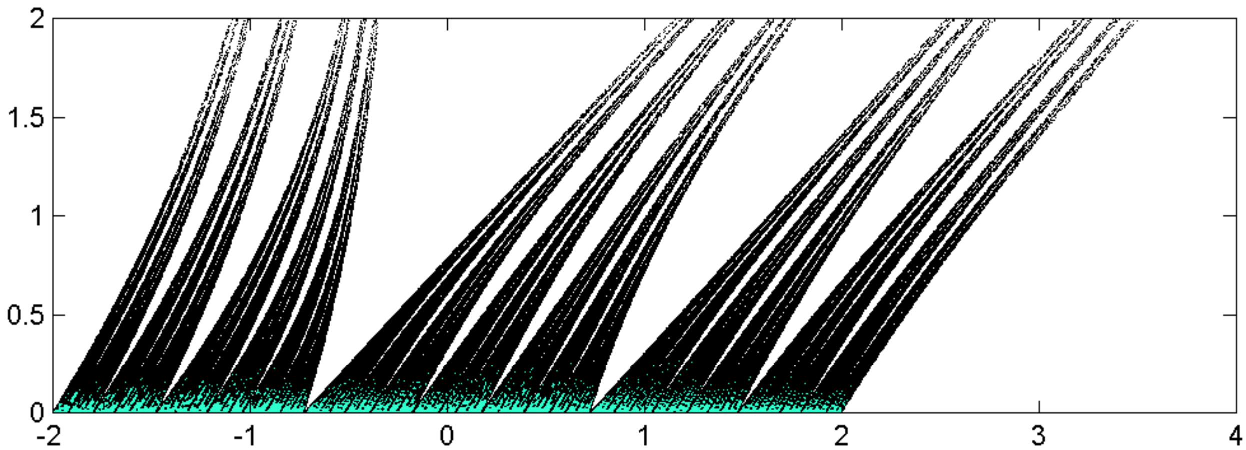

Figure 1: The set $\{(E, V) : E \in \Sigma_V,\ 0 \leq V \leq 2\}$.

Next result concerns the sum set $\Sigma_V + \Sigma_V = \{E_1 + E_2 : E_1, E_2 \in \Sigma_V\}$. This set is equal to the spectrum of the so-called square Fibonacci Hamiltonian. Here, one considers the Schrödinger operator

$$\begin{aligned}[H_V^{(2)}\psi](m,n) = &\psi(m+1,n) + \psi(m-1,n) + \psi(m,n+1) + \psi(m,n-1) + \\ &+ V\left(\chi_{[1-\alpha,1)}(m\alpha \mod 1) + \chi_{[1-\alpha,1)}(n\alpha \mod 1)\right)\psi(m,n)\end{aligned}$$

in $\ell^2(\mathbb{Z}^2)$. This operator and its spectrum have been studied numerically and heuristically by Even-Dar Mandel and Lifshitz (a similar model was studied by Sire). Their study suggested that at small coupling, the spectrum $\Sigma_V + \Sigma_V$ is not a Cantor set; quite on the contrary, it has no gaps at all. In [5] we confirm this observation and prove rigourously that *for sufficiently small coupling, the sum* $\Sigma_V + \Sigma_V$ *is an interval.* Certainly, the same statement holds for the cubic Fibonacci Hamiltonian (i.e., the analogously defined Schrödinger operator in $\ell^2(\mathbb{Z}^3)$ with spectrum $\Sigma_V + \Sigma_V + \Sigma_V$).

Besides, in [5] we prove a version of the Dry Ten Martini Problem for Fibonacci Hamiltonian (i.e. we prove that all labels given by the gap labeling theorem correspond to gaps in the spectrum), describe the rate of the linear gap opening in terms of labels provided by the gap labeling theorem, give explicit upper and lower bounds for the solutions to the associated difference equation, and use them to study the spectral measures and the transport exponents.

In our current work in progress with Damanik we intend to show that as coupling constant increases, the spectrum of the square Fibonacci Hamiltonian bifurcates from an interval to the so called cantorval (compact set with dense interior and without isolated connected components). That will give a new type of spectrum for "natural" potentials.

Conservative homoclinic bifurcations and hyperbolic sets of large Hausdorff dimension

In the case of dissipative dynamical systems on surfaces homoclinic bifurcations were intensively investigated; some of the dynamical phenomena that appear after a bifurcation in this case are persistent tangencies and infinite number of sinks (Newhose phenomena), strange attractors (Mora, Viana), arbitrarily degenerate periodic points of arbitrary high periods (Gonchenko, Shilnikov, Turaev), and superexponential growth of periodic orbits (Kaloshin).

The conservative (area preserving) case is known to be more complicated. For example, it took over two decades to prove an analog of Newhouse results for area preserving surface diffeomorphisms (Duarte). I proved that *locally maximal hyperbolic sets of Hausdorff dimension arbitrary close to two appear after a generic one-parameter unfolding of a homoclinic tangency.* This fact has numerous applications, two of them are presented below.

The results described in this section and in two sections after that will be published in a series of papers. The first paper of the series [6] is published, and the second one [7] is currently submitted for a publication.

On the size of the stochastic layer of the standard map

The KAM theorem on the conservation of quasiperiodic motions in near-integrable Hamiltonian systems gave rise to the question on dynamical behavior in the regions where invariant tori are destroyed. In a more general form this (open) question can be stated in the following way: “Can an analytic symplectic map have a chaotic component of positive measure and the Kolmogorov-Arnold-Moser (KAM) tori coexist? "

The simplest and most famous system where one would expect *mixed behavior* (KAM tori and orbits with non-zero Lyapunov exponents (stochastic sea) both have positive measure) is the Taylor-Chirikov standard map of the two–dimensional torus $\mathbb{T}^2$, given by

$$f_k(x, y) = (x + y + k\sin(2\pi x), y + k\sin(2\pi x)) \bmod \mathbb{Z}^2. \tag{1}$$

I proved that *stochastic layer of the standard map has full Hausdorff dimension for large parameters from a residual set in the space of parameters,* see [6, 7] for the precise statement.

Notice that this result gives a partial explanation of the difficulties that we encounter studying the standard family. Indeed, one of the possible approaches is to consider an invariant hyperbolic set in the stochastic layer and to try to extend the hyperbolic behavior to a larger part of the phase space through homoclinic bifurcations. Unavoidably Newhouse domains associated with absence of hyperbolicity appear after small change of the parameter. If the Hausdorff dimension of the initial hyperbolic set is less than one, then the measure of the set of parameters that correspond to Newhouse domains is small and has zero density at the critical value, as was shown by Newhose-Palis and Palis-Takens. For the case when the Hausdorff dimension of the hyperbolic set is slightly bigger than one, similar result was recently obtained by Palis and Yoccoz, and the proof is astonishingly involved. They also conjectured that analogous property holds for an initial hyperbolic set of any Hausdorff dimension, but the proof would require even more technical and complicated considerations. Here is what Palis and Yoccoz wrote:

“Of course, we expect the same to be true for all cases $0 < \dim_H(\Lambda) < 2$. For that, it seems to us that our methods need to be considerably sharpened: we have to study deeper the dynamical recurrence of points near tangencies of higher order (cubic, quartic, ...) between stable and unstable curves. We also hope that the ideas introduced in the present paper might be useful in broader

contexts. In the horizon lies the famous question whether for the standard family of area preserving maps one can find sets of positive Lebesgue probability in parameter space such that the corresponding maps display non-zero Lyapunov exponents in sets of positive Lebesgue probability in phase space."

The result stated above shows that in order to understand the dynamics of the stochastic layer of the standard map one has to face these difficulties.

Homoclinic bifurcations in some restricted versions of the three body problems

Initially my interest in the conservative bifurcations was motivated by the fact that it appears in the three body problem. The classical three–body problem consists in studying the dynamics of 3 point masses in the plane or in the three-dimensional space mutually attracted under Newton gravitation. The three–body problem is called *restricted* if one of the bodies has mass zero and the other two are strictly positive. In his pioneering work Alexeev found important use of hyperbolic dynamics for the three–body problem. He proved existence of the so called *oscillatory motions*. A motion of the three–body problem is called *oscillatory* if the limsup of the mutual distances is infinite and the liminf is finite. Existence of such motions was a long standing open problem. The first rigorous example of existence of such motions is due to Sitnikov for the restricted spacial three–body problem. Alexeev extended the Sitnikov example to the spatial three–body problem. Later Moser gave a conceptually transparent proof of existence of oscillatory motions for the Sitnikov example interpreting homoclinic intersections. This paved a road to a variety of applications of hyperbolic dynamics to the three–body problem.

A famous open conjecture (that goes back to Kolmogorov or even Chazy) claims that the set of initial conditions that correspond to oscillatory motions has zero measure. In our joint work with Kaloshin we show that *in Sitnikov problem and in the restricted planar circular three body problem invariant hyperbolic sets of large (close to the dimension of the phase space) Hausdorff dimension appear for many values of the parameter*. Then we apply this result to show that in the considered cases *the set of oscillatory motions has full Hausdorff dimension for many values of the parameter*, see [6, 8] for precise statements.

Non-hyperbolic invariant measures for non-hyperbolic homoclinic classes

To what extent is the behavior of a generic dynamical system hyperbolic? A number of problems in modern theory of smooth dynamical systems can be viewed as some forms of this question. It was shown by Abraham and Smale in the 1960s that uniformly hyperbolic systems (Anosov diffeomorphisms, Axiom A) are not dense in the space of dynamical systems. This forced weakening the notion of hyperbolicity and gave rise to notions of partial and nonuniform hyperbolicity. Some systems of these types are studied by famous Pesin theory, where hyperbolic behavior is characterized by nonzero Lyapunov exponents for some invariant measure. The most natural case is that of a system with a smooth invariant measure. However, the question about Lyapunov exponents can also be considered for maps that do not a priori have a natural invariant measure. In particular, it is reasonable to conjecture that *uniformly hyperbolic maps together with diffeomorphisms exhibiting a non-hyperbolic ergodic invariant measure form an open and dense subset in the space of smooth dynamical systems.* In [9] we prove that *this is true for a generic diffeomorphism exhibiting a homoclinic class with saddles of different indices.*

The conjecture above provides a nice description of the space of smooth dynamical systems on ergodic level in a spirit of the famous Palis' Conjectures. In our joint paper with Diaz and Bonatti [10] we prove that *in a partially hyperbolic case (with one-dimensional central bundle) one can guarantee that the support of the constructed non-hyperbolic measure is the whole homoclinic class.* In particular, this explains the difficulties that appear when one is trying to investigate such systems numerically.

Dynamical properties of piecewise isometries

Consider a ball in $\mathbb{R}^n$. Cut the ball into several pieces, and shift each piece in such a way that the image is still inside of the initial ball. Certainly, images of some pieces may overlap. This simple construction defines a (discontinuous) dynamical system with highly non-trivial behavior. Interestingly enough, very little is known about these piecewise isometries. A class of systems of this kind appeared in a natural way in some problems related to machine learning, due to the approach suggested by Max Welling. Currently this promising direction of research is in the beginning stage. Recently we received an NSF grant[2] (joint with Max Welling) to study these systems in details and to apply the results to machine learning.

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