

INTRO DIFFERENTIAL EQUATIONS

Practice Final Exam with answers

Version 1.

Problem 1.

Find a general solution of the equation

$$y' + 2y = y^2 e^x$$

Answer: $y(e^x + Ce^{2x}) = 1; y = 0$

Problem 2.

Find a general solution of the equation

$$y'' + 2y' + y = 3e^{-x}\sqrt{x+1}$$

Answer: $y(x) = e^{-x} \left(\frac{4}{5}(x+1)^{\frac{5}{2}} + C_1 + C_2x \right)$

Problem 3.

Find a general solution of the equation in the domain $x \in (0, \infty)$

$$x^2 y'' - xy' - 3y = 0$$

Answer: $y(x) = C_1 x^3 + C_2 x^{-1}$

Problem 4.

Solve the initial value problem

$$\begin{cases} y' + 2y = f(t) \\ y(0) = 0, \end{cases}$$

where $f(t) = \begin{cases} t, & 0 \leq t < 1; \\ 0, & t \geq 1. \end{cases}$

Answer: $y(x) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} + \left(\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{-2(t-1)}\right)\mathcal{U}(t-1)$

Problem 5.

Solve the initial value problem for the system of differential equations

$$\begin{cases} \frac{d^2x}{dt^2} + x - y = 0 \\ \frac{d^2y}{dt^2} + y - x = 0 \\ x(0) = 0, x'(0) = -2, y(0) = 0, y'(0) = 1 \end{cases}$$

Answer: $x(t) = -\frac{1}{2}t - \frac{3\sqrt{2}}{4}\sin\sqrt{2}t, y(t) = -\frac{1}{2}t + \frac{3\sqrt{2}}{4}\sin\sqrt{2}t$

Version 2.

Problem 1.

Find a general solution of the equation

$$x^2 y' = y(x + y)$$

Answer: $y \ln(Cx) = -x; y = 0$

Problem 2.

Find a general solution of the equation

$$y'' - 5y' = 3x^2 + \sin 5x$$

Answer: $y(x) = C_1 + C_2 e^{5x} - 0.2x^3 - 0.12x^2 - 0.048x + 0.02(\cos 5x - \sin 5x)$

Problem 3.

Find two power series solutions about the point $x = 0$ (it is enough to write down the first several coefficients of the series)

$$y'' - (x + 1)y' - y = 0$$

$$\text{Answer: } \begin{cases} y_1(x) = c_0 \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \dots\right) \\ y_2(x) = c_1 \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \dots\right) \end{cases}$$

Problem 4.

Solve the initial value problem

$$\begin{cases} y'(t) = \cos t + \int_0^t y(\tau) \cos(t - \tau) d\tau \\ y(0) = 1 \end{cases}$$

Answer: $y(t) = 1 + t + \frac{1}{2}t^2$

Problem 5.

Solve the initial value problem for the system of differential equations

$$\begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = 2x \\ x(0) = 0, y(0) = 1 \end{cases}$$

Answer: $x(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t, y(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$

Version 3.

Problem 1.

Find a general solution of the equation

$$y' = \frac{y}{3x - y^2}$$

Answer: $x = Cy^3 + y^2$; $y = 0$

Problem 2.

Find a general solution of the equation

$$y'' - 3y' + 2y = x \cos x$$

Answer: $y(x) = C_1 e^x + C_2 e^{2x} + (0.1x - 0.12) \cos x - (0.3x + 0.34) \sin x$

Problem 3.

Find a general solution of the equation in the domain $x \in (0, \infty)$

$$x^2 y'' - 3xy' + 5y = 3x^2$$

Answer: $y(x) = x^2 (C_1 \cos \ln x + C_2 \sin \ln x + 3)$

Problem 4.

Solve the initial value problem

$$\begin{cases} y' - 3y = \delta(t - 2) \\ y(0) = 0 \end{cases}$$

Answer: $y(t) = e^{3(t-2)} \mathcal{U}(t - 2)$

Problem 5.

Solve the initial value problem for the system of differential equations

$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 5x - y \\ x(0) = -1, y(0) = 2 \end{cases}$$

Answer: $x(t) = -\cos 3t - \frac{5}{3} \sin 3t$, $y(t) = 2 \cos 3t - \frac{7}{3} \sin 3t$