COMPLEX ANALYSIS, HW # 1

Section 9, problem 11, and these problems:

Problem 1.

Give a complete proof of the following. If f is an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$ then f is a polynomial.

Problem 2.

Prove that the order $\lambda(f)$ of an entire function f is given by

$$\lambda = \limsup_{r\to\infty} \frac{\log(\log \|f\|_{\infty,B_r})}{\log r},$$

where $\|f\|_{\infty,B_r} = \sup_{z\in B_r} |f(z)|$.

Problem 3.

Prove that for any increasing function $g : [0, +\infty) \to [0, +\infty)$ there exists an entire function f such that $f(x) > g(|x|)$ for all real values of x.

Problem 4.

Prove that the order $\lambda(f)$ of an entire function $f(z) = \sum_{k=1}^{\infty} a_n z^n$ is given by

$$\lambda = \limsup_{n\to\infty} \frac{n \log n}{-\log |a_n|}$$

Problem 5.

Prove that an entire function f has finite order and no zeros if and only if $f = e^g$ for some polynomial g.

Problem 6.

Give an example of an entire function f of order one with zeroes $\{a_n\}_{n\in\mathbb{N}}$ such that one has $\sum_{n=1}^{\infty} |a_n|^{-1} = \infty$. Check that in your example $\sum_{n=1}^{\infty} |a_n|^{-1-\varepsilon} < \infty$ for any $\varepsilon > 0$.