Ergodic Theory, Homework

Problem 1.

Consider a Lissajous figure on the plane

$$x(t) = A\sin(t+\phi),\ y(t) = B\sin(\omega t+\psi),\ \ t\in\mathbb{R}.$$

Prove that if ω is irrational then for any phases ϕ,ψ this curve is dense in the rectangle $|x|\le A, |y|\le B$.

Problem 2.

Consider the set C of all points of the unit interval which have a binary representation without two successive zeros. Prove that C is an uncountable set invariant under the map $T:[0,1)\to[0,1)$, $T(x)=2x(\mathrm{mod}\ 1)$. Why doesn't it contradict to ergodicity of T?

Problem 3.

Prove that the hyperbolic automorphism of the torus $T:\mathbb{T}^2\to\mathbb{T}^2$ induced by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has infinitely many periodic points, and these periodic points are dense in $\mathbb{T}^2$.

Problem 4.

Prove that the map $T: S^1\to S^1$, $T(x) = 2x(\mathrm{mod}\ 1)$, is a factor (as a topological dynamical system) of the topological Bernoulli shift $\sigma:\Sigma_2\to\Sigma_2$. Also, show that $T:(S^1, Leb)\to(S^1, Leb)$ and the Bernoulli shift $\sigma:(\Sigma_2,\mu_{1/2,1/2})\to(\Sigma_2,\mu_{1/2,1/2})$ are isomorphic (as measure preserving transformations).

Problem 5.

Let $f:[0,1]\to[0,1]$ be a homeomorphism. Prove that $h_{top}(f)=0$.

Problem 6.

Show that $f:[0,1]\to[0,1]$ given by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{when } 0<x\le 1; \\ 1, & \text{when } x=0. \end{cases}$$

has no invariant Borel probability measure. Does it contradict to the Krylov-Bogolyubov Theorem?

Problem 7.

Prove that the map $g:[0,1]\to[0,1]$ given by

$$f(x) = \begin{cases} 2x, & \text{when } 0\le x\le 1/2; \\ 2-2x, & \text{when } 1/2\le x\le 1. \end{cases}$$

preserves Lebesgue measure and is ergodic with respect to Lebesgue measure.

Problem 8.

Consider the map $f : S^1 \times [0,1] \to S^1 \times [0,1]$, $f(x,t) = (x+\alpha, t)$, $\alpha \notin \mathbb{Q}$. Prove that the averages of every continuous function ϕ converge uniformly but that f is not uniquely ergodic.

Problem 9.

Let $\sigma_A : \Sigma_A \to \Sigma_A$ be a topological Markov chain with the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Find $\#Per_n(\sigma_A)$ for each $n \in \mathbb{N}$.

Problem 10.

For each $a > 0$ construct an example of a homeomorphism of a compact metric space with topological entropy a.

Problem 11.

Is it possible to find a topological Markov chain $\sigma_A : \Sigma_A \to \Sigma_A$ such that $h_{top}(\sigma_A) = \ln \pi$?