ERGODIC THEORY, MATH 211A, FINAL

Problem 1.

Calculate the frequency with which 2^n has r ($r = 0, 1, \ldots, 9$) as the second digit of its base 10 representation.

Problem 2.

Let $\beta > 1$ denote the golden ratio (so that $\beta^2 = \beta + 1$). Define

$$T : [0, 1] \to [0, 1], \quad T(x) = \beta x (\text{mod } 1).$$

Shoe that T does not preserve Lebesgue measure. Define

$$\mu(B) = \int_B k(x)dx, \text{ where } k(x) = \begin{cases} \frac{1}{\frac{1}{\beta}+\frac{1}{\beta^3}}, & \text{on } [0, \frac{1}{\beta}); \\ \frac{1}{1+\frac{1}{\beta^2}}, & \text{on } [\frac{1}{\beta}, 1). \end{cases}$$

Prove that T preserves the measure μ.

Problem 3.

Prove that the arithmetic mean of the digits appearing in the base 10 expansion of Lebesgue-a.e. $x \in [0, 1)$ is equal to 4.5, i.e. if $x = 0.x_1x_2x_3\ldots$, $x_i \in \{0, 1, 2, \ldots, 9\}$, then

$$\text{for a.e. } x \in [0, 1) \quad \lim_{n\to\infty} \frac{x_1 + x_2 + \ldots x_n}{n} = 4.5.$$

Problem 4.

Consider $T : \mathbb{T}^2 \to \mathbb{T}^2$, $T(x, y) = (2x, 3y)(\text{mod } 1)$. Prove that T preserves Lebesgue measure. Is it ergodic? Mixing? Is it transitive? Minimal? Find $h_{top}(T)$ and show that Lebesgue measure is the measure of maximal entropy for T.

Problem 5.

Denote the continued fraction expansion of $x \in (0, 1)$ by $x = [a_1, a_2, \ldots]$. Prove that there exists $C > 0$ such that for a.e. $x \in (0, 1)$ we have

$$\lim_{n\to\infty} \frac{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \frac{1}{a_n}}{n} = C.$$