Real Analysis
Math 205C, Spring 2012

HW# 3

Problem 1.

Suppose $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, and $Df(0)$ in not invertible. Does it imply that f is not invertible?

Problem 2.

Prove that for any $n \times n$ matrix A which is close enough to Id_n there exists an $n \times n$ matrix X such that $A = e^X$.

Problem 3.

Is it true that for any $n \times n$ matrix A which is close enough to zero $n \times n$ matrix there exists an $n \times n$ matrix X such that $A = e^X$?

Problem 4.

Consider the equation $xe^y + ye^x = 0$. Show that there is a C^∞ solution $y = y(x)$ of this equation near $(0, 0)$. What is its derivative at $x = 0$? What is its second derivative at $x = 0$?

Problem 5.

Let M denote the space of all $n \times n$ matrices, and G denote the set of invertible $n \times n$ matrices. Prove that inversion $Inv : G \to G, Inv(A) = A^{-1}$ is a diffeomorphism and show that its derivative at A is the linear transformation $M \to M$, $X \mapsto -A^{-1} \circ X \circ A^{-1}$. What does it give you in the case $n = 1$?

Problem 6.

Consider cubic polynomials of the form $f(x) = x^3 + ax^2 + bx + c$, where a, b and c are real. Note that when $a = 1, b = -1$ and $c = 0$ the equation $f(x) = 0$ has three distinct roots, namely, $u = 1$, $v = -1$ and $w = 0$. Use the Inverse Function Theorem to show that when the coefficients (a, b, c) are sufficiently near $(0, -1, 0)$ then the solutions u, v, w of the equation $f(x) = 0$ can be expressed as a continuously differentiable functions of the coefficients a, b, c.

Problem 7.

Define $f : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for x in terms of y (or for y in terms of x).

Problem 8.

Define $f : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for x in terms of y (or for y in terms of x).

Problem 9.

Define $f : \mathbb{R}^3 \to \mathbb{R}^1$ by

$$f(x_1, x_2, y) = y^2 x_1 + e^y + x_2.$$

Show that there exists a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and $f(x_1, x_2, g(x_1, x_2)) = 0$. Find $\frac{\partial g}{\partial x_1}(1, -1)$ and $\frac{\partial g}{\partial x_2}(1, -1)$.

Problem 10.

Let $y = (y_1(x), y_2(x))$ be a function of x defined by a system of equations

$$\begin{cases} y_1^2 + y_2^2 = e^x \\ y_1 = xy_2, \end{cases}$$

where (x, y_1, y_2) is in a neighborhood of the point $\left(1, \sqrt{\frac{e}{2}}, \sqrt{\frac{e}{2}}\right)$. Compute $\frac{dy_1}{dx}$ at the point $\left(1, \sqrt{\frac{e}{2}}, \sqrt{\frac{e}{2}}\right)$. Justify the existence of the derivative.