Real Analysis
Math 205C, Spring 2012

HW# 2

In Problems 1 - 4 calculate all first-order partial derivatives of the function $f : \mathbb{R}^n \to \mathbb{R}^1$, if

Problem 1.

$f(x) = a \cdot x$, where a is a fixed vector in $\mathbb{R}^n$;

Problem 2.

$f(x) = \|x\|^4$;

Problem 3.

$f(x) = x \cdot L(x)$, where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function;

Problem 4.

$f(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$, where $a_{ij} = a_{ji}$.

Problem 5.

Suppose $f \in C^1(\mathbb{R}^2, \mathbb{R}^1)$. Let $F(r, \theta) = f(r\cos\theta, r\sin\theta)$.
Calculate $\frac{\partial F}{\partial r}$ and $\frac{\partial F}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Problem 6.

Suppose $f \in C^1(\mathbb{R}^2, \mathbb{R}^1)$. Let $F(r, \theta) = f(r\cos\theta, r\sin\theta)$. Show that

$$\|\nabla f(r\cos\theta, r\sin\theta)\|^2 = \left(\frac{\partial F}{\partial r}(r,\theta)\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \theta}(r,\theta)\right)^2$$

Problem 7.

Suppose $U \subset \mathbb{R}^n$ is open and connected, $f : U \to \mathbb{R}^m$ is second differentiable everywhere and $(D^2 f)_p = 0$ for all $p \in U$. What can you say about the function f?

Problem 8.

Let f be a C^1 function from the interval $(-1, 1)$ into $\mathbb{R}^2$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that there is a number $\varepsilon \in (0, 1)$ such that $\|f(t)\|$ is an increasing function of t on $(0, \varepsilon)$.

Problem 9.

Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^1$, $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$
has well defined partial derivatives everywhere, but is not continuous at the origin.

Problem 10.

Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^1$, $f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$
has well defined partial derivatives and is continuous everywhere, but is not differentiable at the origin.