Real Analysis
Math 205C, Spring 2012

HW# 1

Problem 1.

Two norms, $|\cdot|_1$ and $|\cdot|_2$, on a vector space V are equivalent if there are positive constants C_1, C_2 such that for any nonzero vector $v \in V$

$$C_1 \leq \frac{|v|_1}{|v|_2} \leq C_2.$$

Prove that this gives an equivalence relation on norms.

Problem 2.

Prove that any two norms on a finite-dimensional vector space are equivalent. *(Hint: We did that last quarter!)*

Problem 3.

Consider the norms

$$|f|_1 = \int_0^1 |f(t)|dt, \quad \text{and} \quad |f|_{C^0} = \max\{|f(t)| \ : \ t \in [0,1]\}$$

defined on $C^0[0,1]$. Show that these norms are not equivalent.

Problem 4.

Prove that for any operators $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ we have $\|AB\| \leq \|A\| \cdot \|B\|$. Give an example of two 2×2 matrices such that the norm of the product is strictly less than the product of the norms.

Problem 5.

Consider the matrix $S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ it represents. Calculate the norm of S.

Problem 6.

Prove that the set of invertible $n \times n$ matrices is open in the space of all $n \times n$ matrices. Is it dense?

Problem 7.

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(s,t) = (x,y,z)$, where $x(s,t) = st, y(s,t) = s\cos t, z(s,t) = s\sin t$, and $g(x,y,z) = xy + yz + zx$. Calculate the derivative of $g \circ f$ directly and then using the Chain Rule.

Problem 8.

Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is differentiable at 0 and discontinuous at any other point.

Problem 9.

If f and g are differentiable real valued functions in $\mathbb{R}^n$, prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

Problem 10.

Suppose f is differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^3$ such that $|f(t)| = 1$ for every t. Prove that $f'(t) \cdot f(t) = 0$. Interpret this result geometrically.