Real Analysis
Math 205B, Winter 2012

HW# 2

Problem 1.

Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & \text{if } x \le 0; \\ x^{\frac{1}{x}}, & \text{if } x > 0. \end{cases}$$

Prove (provide the details!) that $f \in C^\infty(\mathbb{R})$, and also calculate explicitly f' and f''.
Hint: compare with Lemma 11.10 from Carothers.

Problem 2.

Write an explicit formula for some C^∞ function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = \sin x$ if $x > 100$, and $g(x) = \cos x$ if $x < -100$.

Problem 3.

Set

$$R(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}; \\ \frac{1}{m}, & \text{if } x = \frac{n}{m},\ n \in \mathbb{Z},\ m \in \mathbb{N},\ (n,m) = 1. \end{cases}$$

Is it possible for some differentiable function $g : \mathbb{R} \to \mathbb{R}$ to have a derivative $g' = R$? Explain.

Problem 4.

Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}, \quad (a < x < b).$$

Problem 5.

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function, with bounded derivative (say, $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. Provide an explicit range of admissible values of ε (that should depend only on M).

Problem 6.

If

$$C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \ldots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Problem 7.

Suppose f is defined and differentiable for every $x > 0$, and $f' \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

Problem 8.

Suppose f is defined and twice differentiable for every $x > 0$, f'' is bounded on $(0, +\infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Problem 9.

Call x a *fixed point* of f if $f(x) = x$. If $f : \mathbb{R} \to \mathbb{R}$ differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Problem 10.

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that at some point $x \in \mathbb{R}$ the limit

$$\lim_{h \to 0} \frac{f(x-h) + f(x+h) - 2f(x)}{h^2}$$

exists, but $f''(x)$ fails to exist.