Midterm Sample

Problem 1.

Prove that the following improper integral converges: $\int_0^\infty \frac{\cos(2x)}{x^{1/3}} dx$

Problem 2.

Suppose U is an open subset of $\mathbb{R}$, containing a point x_0, f and g are real-valued functions, defined on U, such that g is continuous, f is differentiable, and $f(x_0) = 0$. Prove that the froduct fg is differentiable at x_0.

Problem 3.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is such that f, f', and f'' are all bounded and continuous real-valued functions. Assume also that $f(0) = f'(0) = 0$. Analyze the convergence of the series $\sum_{n=1}^{\infty} f\left(\frac{x}{n}\right)$, i.e. determine for which values of x the series is convergent, and for which values of x the series converges absolutely. Is the convergence uniform? Justify all your claims.

Problem 4.

Determine whether the family of functions $f_\alpha : [0,1] \to \mathbb{R}$, given by the formula

$$f_\alpha(x) = \frac{1}{1+e^{\alpha x}} \quad \text{for } \ x \in [0,1], \alpha \in [1,\infty),$$

is equicontinuous on $[0,1]$. Justify your answer.

Problem 5.

Let $f_n, n = 1, 2, \ldots$ and f be Riemann integrable real-valued functions defined on $[0,1]$. For each of the following statements, determine whether the statement is true or not:

$$(a) \ \text{If} \ \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)| dx = 0, \ \text{then} \ \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 dx = 0.$$

$$(b) \ \text{If} \ \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 dx = 0, \ \text{then} \ \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)| dx = 0.$$