REAL ANALYSIS
MATH 205A, FALL 2011

Final Exam

Thursday, November 3, 2011 — 10:0-10:50am

Problem	1	2	3	4	5	Σ
Points						

Student's name:

Problem 1.

Prove that

$$\liminf_{n\to\infty}(a_n - b_n) \geq \liminf_{n\to\infty} a_n - \limsup_{n\to\infty} b_n$$

for any real sequences $\{a_n\}$, $\{b_n\}$.

Problem 2.

Let Σ^2 be the space of all infinite sequences of 0's and 1's,

$$\Sigma^2 = \{\bar{\omega} \mid \bar{\omega} = \omega_1\omega_2\omega_3\ldots\omega_k\ldots, \omega_i \in \{0,1\}\}.$$

Show that

$$d(\bar{\omega}, \bar{u}) = \begin{cases} 0, & \text{if } \bar{\omega} = \bar{u}; \\ 2^{-m}, & \text{if } \omega_i = u_i \text{ for } i < m \text{, but } \omega_m \neq u_m \end{cases}$$

is a metric on Σ^2.

Problem 3.

Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Does it imply that the series

$$\sum_{n=1}^{\infty}(a_n + a_n^2 + \ldots + a_n^n)$$

converges?

Problem 4.

Fix $k \in \mathbb{N}$ and define $f : l_1 \to \mathbb{R}$ by $f(x) = x_k$ (here $x = (x_1, \ldots, x_k, \ldots) \in l_1$). Show that f is continuous.

Problem 5.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $|x| = \left(x_1^2 + \ldots + x_n^2\right)^{1/2}$. Suppose f maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ in such a way that $|f(x) - f(y)| \geq |x - y|$ for any $x, y \in \mathbb{R}^2$. Suppose A is an open subset of $\mathbb{R}^n$. Prove that $f(A)$ is open.