Real Analysis
Math 205A, Fall 2011

Final Sample

Problem 1.

Suppose A is a subset of a complete metric space (M, d), and f is a uniformly continuous M-valued function, defined on A. Prove that there exists a uniformly continuous function $g : \overline{A} \to M$ such that $g|_A = f$.

Problem 2.

Suppose that $a_n > 0$ and $\sum a_n$ diverges. Prove that $\sum \frac{a_n}{1+a_n}$ also diverges.

Problem 3.

Let $E \subset \mathbb{R}^n$ be a non-compact set. Prove that there exists a bounded continuous function $f : E \to \mathbb{R}$ that has no maximum value.

Problem 4.

Let $\mathbf{P}$ be the vector space of all polynomials supplied with the norm $\|p\| = \max\{|a_i| \mid i = 0, 1, \ldots, deg(p)\}$, where $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathbf{P}, n = deg(p)$. Show that $\mathbf{P}$ is not complete.

Problem 5.

Let X be a metric space. A function $f : X \to \mathbb{R}$ is called *lower semicontinuous* if

$$f^{-1}((a, \infty)) \text{ is open for any } a \in \mathbb{R}.$$

Show that

$$\liminf_{n\to\infty} f(x_n) \geq f(x_0)$$

whenever x_n is a sequence in X with $\lim_{n\to\infty} x_n = x_0$ if f is lower semicontinuous.