ABSORBING BOUNDARY CONDITIONS FOR DOMAIN DECOMPOSITION

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Abstract. In this paper we would like to point out some similarities between two artificial boundary conditions. One is the far field or absorbing boundary conditions for computations over unbounded domain. The other is the boundary conditions used at the boundary between subdomains in domain decomposition. We show some convergence result for the generalized Schwarz alternating method (GSAM), in which a convex combination of Dirichlet data and Neumann data is exchanged at the artificial boundary. We can see clearly how the mixed boundary condition and the relative size of overlap will affect the convergence rate. These results can be extended to more general coercive elliptic partial differential equations using the equivalence of elliptic operators. Numerically first and second order approximations of the Dirichlet to Neumann operator are constructed using local operators, where information tangential to the boundary is included. Some other possible extensions and applications are pointed out. Finally numerical results are presented.

Key Words. partial differential equation (PDE), domain decomposition, Schwarz alternating methods (SAM), absorbing boundary condition, Dirichlet to Neumann (DtN) operator, immersed interface problem, trace theorem.

1. Introduction.

There are some similarities in the requirements for two classes of artificial boundary conditions. One consisting of those conditions used in computer simulations over unbounded domains and the other of those used in domain decomposition. In both cases the original problem does not contain these artificial boundary conditions and the differential equation is valid across the location of the artificial boundaries. The boundary condition should thus be as "transparent" as possible for true information and not introduce significant errors. Usually some local approximations of some sort of Dirichlet to Neumann (DtN) map are used at the artificial boundary. In this paper we will show both the similarities and differences between these classes of boundary conditions. We will also describe how techniques which were developed in simulation over unbounded domains can be beneficial as boundary conditions in domain decomposition.

We shall use the term absorbing boundary condition (ABC) for the artificial boundary condition which is used to limit the computational domain in simulations over unbounded domains. Depending on application, these types of boundary conditions go under many names, for example, radiation, far field or open.

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The basic requirements for ABCs are, *stability, accuracy and computational efficiency.* First of all, well posedness of the initial boundary value problem and the stability of the corresponding computational algorithm is clearly necessary. With accuracy, we here mean that the true solution should well satisfy the ABC. The stability will then guarantee a good approximation over all. The ABC should also be computationally efficient. This is, for example, why local ABCs are often preferred over global ones, at least for time dependent problems.

In domain decomposition the computational domain is subdivided into different parts which are connected by artificial boundary conditions. The purpose may be to reduce the computational complexity or to facilitate parallel processing. We shall here consider the iterative Schwarz alternating method (SAM) for elliptic problem. It is also possible to study time dependent problems in terms of overlapping grids.

In domain decomposition the subproblems with their boundary conditions must be individually well posed. Computational efficiency is also important. Accuracy at the artificial boundary, however, plays a slightly different role compared to ABC. The solution is not expected to satisfy the artificial boundary condition directly but only in the limit of an iterative process. The source term in the artificial boundary condition will be iteratively adjusted to achieve accuracy. The standard methodology is to use Dirichlet or Neumann conditions, in which errors are damped out in iterations due solely to the overlap. We will show that "better" boundary conditions will give faster convergence, in particular, if the overlaps between the domains are small. Better boundary conditions may include derivatives both normal and tangential to the boundaries, as is common for ABCs. In the spirit of ABCs, this 'better' boundary condition will introduce an extra cancellation (absorbing) effect at the artificial boundary between subdomains and prevent error from passing from one subdomain to other subdomains freely even with no overlap at all. All of these analysis have used the ellipticity of the PDE. However the idea can be extended to more general situations.

2. Schwarz Alternating Method. The classical SAM method is an iterative method which solves the subproblem in each subdomain alternately and couple the subproblems together by overlapping each subdomain and exchanging function value (Dirichlet data) at the artificial boundaries. The classical SAM method is convergent for elliptic PDEs and its convergence rate is related to the size of overlap of the subdomains. This can be easily seen by either maximum principle or variational interpretation. (see [20],[13],[14]).

Recently many new variants of classical SAM have been proposed for domain decomposition method with or without overlap. (see [3],[1],[4],[16]) Our paper is motivated by the new modified SAM method, proposed by P.L.Lions ([15]), in which there are exchanges of a convex combination of Neumann and Dirichlet data at the artificial boundaries. Convergence was proved in a very general setting with no overlap. The proof is based on delicate energy estimates and Sobolev inequalities. However no estimates on the convergence has been done. Here we will study convergence properties for even more general artificial boundary conditions. We will see improved convergence properties due to the 'positive' property of the DtN operator. (See definition1 below.)

In [23] some analysis was done in linear algebra context and in [22] some analysis was done on the discretized equation. In our paper we are mainly interested in the better understanding of the effect of boundary conditions at the artificial boundary which is also useful in other situations such as overlapping grids and interface problems. In many practical applications, domain decomposition method has also been used as a preconditioner combined with a Krylov subspace method (see [21]), which could be a potential use for the techniques discussed here.

An interesting application of the generalized SAM is to interface problems where the solutions to the elliptic PDE and their derivatives may be discontinuous across some interface due to the discontinuities in the coefficients or singular sources along the interface. This kind of interface problems occur very often in e.g. multiphase or free surface fluid problems, phase transition problems and immersed boundary problems. A domain decomposition along the interface is very natural since we can solve well behaved problems in each subdomain using any good numerical scheme and only couple the subproblems together by the prescribed interface conditions. In this situation the usual elliptic solver can not work well due to the discontinuities across the interface where the PDE is not defined. Even the traditional non-overlapping method in domain decomposition will not work since the Steklov-Poincaré operator or the Schur complement at the interface is not clear. Some brief discussion and references will be presented at the end of section4.

For simplicity we first define the generalized SAM in the case where the whole domain Ω is decomposed into two subdomains Ω_1 and Ω_2 (i.e. $\Omega = \Omega_1 \cup \Omega_2$) with the two artificial boundaries Γ_1, Γ_2 intersecting $\partial\Omega$. Denote $\Omega_{11} = \Omega_1 \setminus \Omega_2, \Omega_{22} = \Omega_2 \setminus \Omega_1$ if there is overlap. (see figure 1)

The elliptic PDE with Dirichlet boundary data is:

$$\begin{aligned} \mathcal{L}u &= f \quad in\ \Omega \\ u &= g \quad on\ \partial\Omega \end{aligned} \tag{1}$$

where $\mathcal{L}$ is some elliptic differential operator. We generalize the classical SAM method to the following two formulations.

(i) additive version.

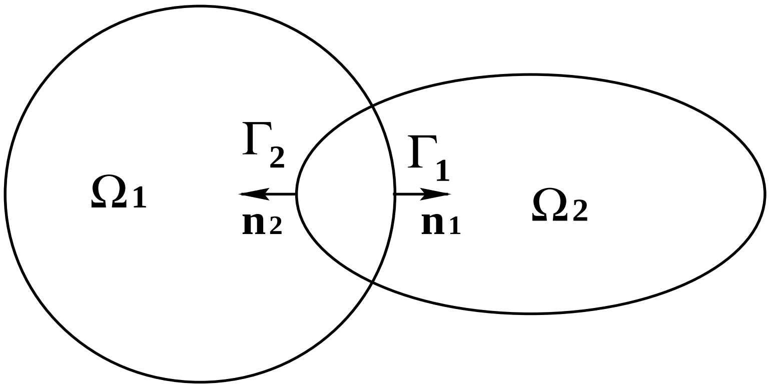


FIG. 1. *Domain decomposition with two subdomains.*

Set u_i^0 initially in Ω_i, $i = 1, 2$ and construct u_i^n in parallel

$$
(2) \qquad \begin{aligned}
\mathcal{L}u_1^n &= f && \text{in } \Omega_1 \\
u_1^n &= g && \text{on } \partial\Omega_1 \backslash \Gamma_1 \\
\Lambda_1 u_1^n + \lambda_1 \frac{\partial u_1^n}{\partial n_1} &= \Lambda_1 u_2^{n-1} + \lambda_1 \frac{\partial u_2^{n-1}}{\partial n_1} && \text{on } \Gamma_1
\end{aligned}
$$

and

$$
(3) \qquad \begin{aligned}
\mathcal{L}u_2^n &= f && \text{in } \Omega_2 \\
u_2^n &= g && \text{on } \partial\Omega_2 \backslash \Gamma_2 \\
\Lambda_2 u_2^n + \lambda_2 \frac{\partial u_2^n}{\partial n_2} &= \Lambda_2 u_1^{n-1} + \lambda_2 \frac{\partial u_1^{n-1}}{\partial n_2} && \text{on } \Gamma_2 \\
& && n = 1, 2, \ldots,
\end{aligned}
$$

(ii) multiplicative version.
Set u_i^0 initially in Ω_i and construct u_1^{2k+1} in Ω_1 and u_2^{2k+2} in Ω_2 sequentially,

$$
(4) \qquad \begin{aligned}
\mathcal{L}u_1^{2k+1} &= f && \text{in } \Omega_1 \\
u_1^{2k+1} &= g && \text{on } \partial\Omega_1 \backslash \Gamma_1 \\
\Lambda_1 u_1^{2k+1} + \lambda_1 \frac{\partial u_1^{2k+1}}{\partial n_1} &= \Lambda_1 u_2^{2k} + \lambda_1 \frac{\partial u_2^{2k}}{\partial n_1} && \text{on } \Gamma_1
\end{aligned}
$$

and

$$
(5) \qquad \begin{aligned}
\mathcal{L}u_2^{2k+2} &= f && \text{in } \Omega_2 \\
u_2^{2k+2} &= g && \text{on } \partial\Omega_2 \backslash \Gamma_2 \\
\Lambda_2 u_2^{2k+2} + \lambda_2 \frac{\partial u_2^{2k+2}}{\partial n_2} &= \Lambda_2 u_1^{2k+1} + \lambda_2 \frac{\partial u_1^{2k+1}}{\partial n_2} && \text{on } \Gamma_2 \\
& && k = 0, 1, 2, \ldots,
\end{aligned}
$$

The solution u^n in Ω can be composed in many ways from u_1^n and u_2^n such that $u^n \in H^1(\Omega) \cap C(\Omega)$ and, $u^n = u_i^n$, in $\Omega_i \setminus (\Omega_1 \cap \Omega_2)$. Λ_i's are some operators and λ_i's are constants. It is easily seen that if $\Lambda_1 = \Lambda_2 = I$(identity operator) and $\lambda_1 = \lambda_2 = 0$, then the above multiplicative version becomes the classical SAM. If $\Lambda_1 = \Lambda_2 = constant > 0$ and $\lambda_1 = \lambda_2 = 1$ then it is the modified SAM proposed by P.L. Lions in [15]. We can also change Λ_i, λ_i's with every iteration. This will give us an accelerated version of the generalized SAM which is discussed briefly in section4. In particular if the Dirichlet data and Neumann data are exchanged alternately then we get the Dirichlet to Neumann type or the relaxation procedure in [16]. Many variants of the classical SAM method can be regarded as a particular choice of Λ_i and λ_i. Without loss of generality we will normalize λ_i's to be 1 if they are not zero, i.e. if the generalized SAM does not degenerate to classical SAM.

Although the practical implementation of the additive scheme and the multiplicative scheme can be quite different, the convergence analysis in two subdomain case is about the same. We will mainly do analysis on the additive scheme for easier book-keeping. The corresponding results for multiplicative scheme will be mentioned without proof and the differences will be pointed out.

At the artificial boundaries in our generalized SAM, a generalized Robin type of boundary condition is imposed. In order to have well-posedness for each subdomain problem, we need some requirement on the operators $\Lambda_i, \ i = 1, 2$

DEFINITION 1. *Let γ_0 be an open (relative to $\partial\Omega$) subset of $\partial\Omega$. A bounded linear operator $\Lambda : H^{1/2}(\gamma_0) \to H^{-1/2}(\gamma_0)$ is said to be positive if for*

$$\forall u \qquad \int_{\gamma_0} u\Lambda u \geq 0$$

It is said to be strictly positive if $\int_{\gamma_0} u\Lambda u > 0, \ if \ u \neq 0$.

Let $\Lambda(x, \eta)$ denote the symbol of the pseudo-differential operator Λ. If Λ is (strictly) positive then

$$\Lambda(x, \eta) \geq (>)0, \qquad \forall x \in \gamma_0, \ \forall\eta$$

The identity operator $\Lambda u = u|_{\gamma_0}$ is a trivial strictly positive operator. The DtN operator for the Laplace equation

$$-\Delta u = 0, \ in\Omega \qquad u = 0 \ on \ \partial\Omega \setminus \gamma_0$$

$\Lambda u = \dfrac{\partial u}{\partial n}|_{\gamma_0}$ is a positive operator. This is also true for any second order coercive PDE.

It is very easy to see the following lemma for the well-posedness of the two sub-problems in $\Omega_1, \ \Omega_2$

LEMMA 2.1. *If* $\Lambda_i : H^{1/2}(\partial\Omega_i \setminus \partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega_i \setminus \partial\Omega),\ i = 1, 2$ *are positive operators, then the two boundary value problems (2),(3) and (4),(5) are well-posed in* H^1.

Theoretically the perfect choice of Λ_i's are the corresponding DtN operators which gives the convergence in two steps even with no overlap. It can be seen through the following lemma.

LEMMA 2.2. *A second order coercive elliptic PDE* $\mathcal{L}u = f$ *in* Ω *can be solved by the generalized SAM without overlap in two steps if* Λ_1*(or* Λ_2*) is the Dirichlet to Neumann operator at the artificial boundary* Γ_1*(or* Γ_2*) for the corresponding homogeneous PDE in* Ω_2*(or* Ω_1*) with homogeneous boundary condition on* $\partial\Omega_2 \cap \partial\Omega$*(or* $\partial\Omega_1 \cap \partial\Omega$*). (see figure 1)*

Proof: We will show that in two steps we can get the exact solution u in Ω. Let $e_i^n = u - u^n,\ i = 1, 2$ be the error function. Then e_i^n will satisfy the following homogeneous equation and boundary condition,

$$\begin{cases} \mathcal{L}e_1^1 = 0 & in\ \Omega_1 \\ e_1^1 = 0 & on\ \partial\Omega_1 \cap \partial\Omega \\ \dfrac{\partial e_1^1}{\partial n_1} + \Lambda_1 e_1^1 = \dfrac{\partial e_2^0}{\partial n_1} + \Lambda_1 e_2^0 & on\ \Gamma_1,\ where\ n_1\ is\ the\ outer\ normal\ for\ \Omega_1 \end{cases}$$

Similarly we have the following problem in Ω_2,

$$\begin{cases} \mathcal{L}e_2^1 = 0 & in\ \Omega_2 \\ e_2^1 = 0 & on\ \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial e_2^1}{\partial n_2} + \Lambda_2 e_2^1 = \dfrac{\partial e_1^0}{\partial n_2} + \Lambda_2 e_1^0 & on\ \Gamma_2,\ where\ n_2\ is\ the\ outer\ normal\ for\ \Omega_2 \end{cases}$$

Since Λ_1 (Λ_2) is the DtN operator at Γ_1,(Γ_2) in Ω_2,(Ω_1), we must have,

$$\frac{\partial e_2^0}{\partial n_1} + \Lambda_1 e_2^0 = -\frac{\partial e_2^0}{\partial n_2} + \Lambda_1 e_2^0 = 0$$

$$\left(\frac{\partial e_1^0}{\partial n_2} + \Lambda_2 e_1^0 = -\frac{\partial e_1^0}{\partial n_1} + \Lambda_2 e_1^0 = 0\right)$$

and we get $e_1^1 = 0$ *in* Ω_1 ($e_2^1 = 0$ *in* Ω_2). Hence we get the exact solution in two steps. □

Remark: This result is essentially the same as the exact non-reflecting boundary conditions in [10]. In general the DtN operator is a global operator and to find its action on u is equivalent to solving a Dirichlet problem. Here we are going to use a positive operator to approximate the DtN operator at the artificial boundary. Due to the positivity of DtN operator for strictly elliptic PDEs, we will show convergence through iterations. For ordinary differential equations with constant coefficients DtN operator can be easily found, (see [24]). For more general elliptic PDEs in simple geometries it is possible to incorporate this into the solving procedure by appropriate finite element formulation, (see [9]).

Since we are interested in the convergence, we only need to look at error function sequence $e_i^n = u - u_i^n,\ i = 1,2$ which satisfies the corresponding homogeneous equation and homogeneous boundary conditions at $\partial\Omega \cap \partial\Omega_i$. From now on we shall assume that $f = 0,\ g = 0$ in (1) and analyze how u_i^n converges to 0 in $\Omega_i,\ i = 1,2$. In the rest of our paper we use two special problems in section3 to analyze the convergence properties of the generalized SAM and generalize the analysis to more general situations in section4. In section5 numerical schemes using local operators to approximate the DtN operator are derived and analyzed. Finally some numerical results are shown in section6.

3. Convergence Analysis of the Generalized SAM for two model problems. In this section we will use Fourier analysis for the Laplace equation in two special geometries. These two model problems let us see very intuitively why the generalized SAM works and how the convergence rate depends on Λ_i, the size of overlap and the size of the subdomains. This is possible since for these two model problems the explicit formula of the convergence rate can be derived. Using the spectral equivalence of the elliptic differential operators we will see that the analysis for these two simple model problems can be generalized to more general elliptic PDEs and geometries in the case of two subdomains.

Model Problem 1:

$$\begin{cases} -\triangle u = f & in\ \Omega \\ \quad u = g & on\ \partial\Omega \end{cases}$$

$$\begin{aligned}
&\Omega = \{(x,y)| - l_1 \le x \le l_2, 0 \le y \le 2\pi\} \\
&\Omega_1 = \{(x,y)| - l_1 \le x \le \delta, 0 \le y \le 2\pi\},\ \Omega_2 = \{(x,y)| - \delta \le x \le l_2, 0 \le y \le 2\pi\} \\
&0 \le \delta < min(l_1, l_2) \\
&\Gamma_1 = \{(x,y)|x = \delta,\ 0 \le y \le 2\pi\},\ \Gamma_2 = \{(x,y)|x = -\delta,\ 0 \le y \le 2\pi\},
\end{aligned}$$

THEOREM 3.1. *For model problem 1 (see figure 2) the generalized additive Schwarz alternating method, (2) - (3). ($\lambda_1 = \lambda_2 = 1$) is convergent if $\Lambda_1 = \Lambda_2 = \Lambda$ is strictly positive. If there is overlap, then there is a constant $0 < \rho < 1$ such that*

$$\| u - u_i^n \|_{H^1(\Omega_i)} \le \rho \| u - u_i^{n-2} \|_{H^1(\Omega_i)} \qquad i = 1,2$$

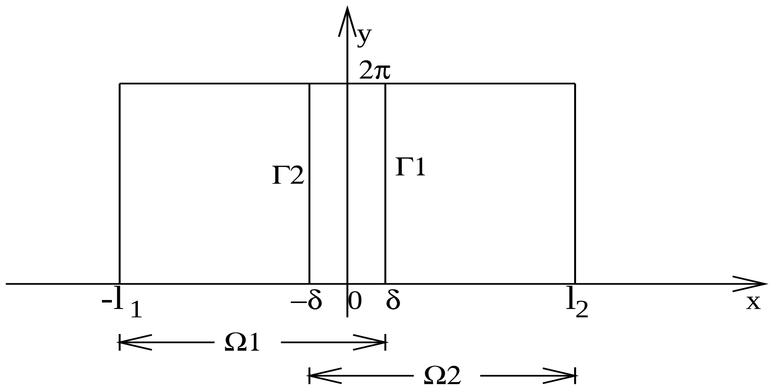


FIG. 2. *Rectangular domain with both subdomains touching the boundary.*

where ρ depends on Λ, the size of $\Omega_i(l_i)$ and the size of overlap δ. If there is no overlap ($\delta = 0$) then

$$\lim_{n\to\infty} \| u - u_i^n \|_{H^1(\Omega_i)} \to 0 \qquad i = 1,2$$
$$\lim_{n\to\infty} \| u - u_i^n \|_{H^{1/2}(\Gamma_i)} \to 0 \quad i = 1,2$$

Proof: Without loss of generality we assume $f = g = 0$. After Fourier transform in y in Ω_1 Ω_2, we have,

$$-\hat{u}_{i_{xx}}^{n}(x,\eta) + \eta^2 \hat{u}_i(x,\eta) = 0, \;\; i = 1,2 \tag{6}$$

where $\hat{u}_i$ is the Fourier transform of u_i, η is the Fourier mode for y. After multiplying both sides by $\overline{\hat{u}_i}$ and integrating by parts we have the following relation,

$$\begin{aligned} \int_{-l_1}^{\delta} \mid \hat{u}_{1_x}^{\,n} \mid^2 + \eta^2 \mid \hat{u}_1{}^n \mid^2 dx &= Re\overline{\hat{u}_1{}^n}\hat{u}_{1_x}^{\,n} \mid_{x=\delta} \\ \int_{-\delta}^{l_2} \mid \hat{u}_{2_x}^{\,n} \mid^2 + \eta^2 \mid \hat{u}_2{}^n \mid^2 dx &= -Re\overline{\hat{u}_2{}^n}\hat{u}_{2_x}^{\,n} \mid_{x=-\delta} \end{aligned} \tag{7}$$

These two relations are the Fourier transform of the following identity

$$\int_\Omega \mid \nabla u \mid^2 = \int_{\partial\Omega} u \frac{\partial u}{\partial n}$$

i.e. the H^1 seminorm of solution is bounded by some boundary integral and the DtN operator is positive.

From (6) and the homogeneous boundary condition on $\partial\Omega$ we have,

$$\hat{u}_1{}^n(x,\eta) = A_1^n(\eta)(e^{\lambda x} - e^{-2\lambda l_1}e^{-\lambda x}) \;\; x \in [-l_1, \delta]$$
$$\hat{u}_2{}^n(x,\eta) = A_2^n(\eta)(e^{-\lambda x} - e^{-2\lambda l_2}e^{\lambda x}) \;\; x \in [-\delta, l_2]$$

where $\lambda(\eta) = |\eta|, \eta = \pm1, \pm2, \ldots$, which is the Fourier transform of DtN operator for our PDE if $l_i = \infty,\ i = 1, 2$. According to equation (7), we have

$$
\begin{aligned}
&\int_{-l_1}^{\delta} \mid \hat{u}_{1x}^{n} \mid^2 + \eta^2 \mid \hat{u}_1^{\,n} \mid^2 dx = \lambda \mid A_1^n(\eta) \mid^2 (e^{2\lambda\delta} - e^{-4\lambda l_1} e^{-2\lambda\delta}) \\
&\int_{-\delta}^{l_2} \mid \hat{u}_{2x}^{n} \mid^2 + \eta^2 \mid \hat{u}_2^{\,n} \mid^2 dx = \lambda \mid A_2^n(\eta) \mid^2 (e^{2\lambda\delta} - e^{-4\lambda l_2} e^{-2\lambda\delta})
\end{aligned} \tag{8}
$$

Denote $\Lambda(\eta)$ to be the Fourier transform (pseudo-differential operator) of the operator Λ. At Γ_1,

$$\hat{u}_{1x}^{n} + \Lambda(\eta)\hat{u}_1^{\,n} \mid_{x=\delta} = A_1^n(\eta)[(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}]$$

$$\hat{u}_{2x}^{n} + \Lambda(\eta)\hat{u}_2^{\,n} \mid_{x=\delta} = -A_2^n(\eta)[(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}]$$

So if we impose the generalized boundary condition at Γ_1 we have

$$\frac{A_1^n(\eta)}{A_2^{n-1}(\eta)} = -\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}} \tag{9}$$

From the boundary condition at Γ_2, similarly we have,

$$\frac{A_2^n(\eta)}{A_1^{n-1}(\eta)} = -\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_1}e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_2}e^{-\lambda(\eta)\delta}} \tag{10}$$

So the convergence rate for $A_i^n,\ i = 1, 2$ is:

$$r_G(\eta, \delta, l_1, l_2, \Lambda) \stackrel{def}{=} \left|\frac{A_1^n(\eta)}{A_1^{n-2}(\eta)}\right| = \left|\frac{A_2^n(\eta)}{A_2^{n-2}(\eta)}\right|$$

$$
\begin{aligned}
&= \sqrt{\left|\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_2}e^{-\lambda(\eta)\delta}}\right|} \\
&\times \sqrt{\left|\frac{(\lambda(\eta) - \Lambda(\eta))e^{-\lambda(\eta)\delta} + (\lambda(\eta) + \Lambda(\eta))e^{-2\lambda(\eta)l_1}e^{\lambda(\eta)\delta}}{(\lambda(\eta) + \Lambda(\eta))e^{\lambda(\eta)\delta} + (\lambda(\eta) - \Lambda(\eta))e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}}\right|}
\end{aligned} \tag{11}
$$

Now we observe the following two facts for the two fractions in (11):

(i) two terms in the numerator and in the denominator have the same product.
(ii) The term with the largest absolute value and the term with the smallest absolute value are in the denominator, since $\delta < min(l_1, l_2)$, and $\Lambda(\eta)$ is strictly positive.

From the simple algebraic fact we know that the norm of each fraction is less than 1.

$$r_G(\eta, \delta, l_1, l_2, \Lambda) < 1 \quad \forall \eta \ and\ \delta \geq 0 \tag{12}$$

Even if $\delta = 0$, i.e. there is no overlap, $r_G(\eta, \delta, l_1, l_2, \Lambda) < 1$. r_G is a decreasing function of δ, because the larger the δ the bigger the difference between the two terms in the denominator and the product is fixed. If there is overlap, i.e. $\delta > 0$,

$$\lim_{|\eta|\to\infty} r_G(\eta, \delta, l_1, l_2, \Lambda) = \left|\frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)}\right| e^{-2\lambda(\eta)\delta} \to 0$$

since $\Lambda(\eta)$ is a pseudo-differential operator from $H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$. Let

$$\sup_{\eta} \; r_G(\eta, \delta, l_1, l_2, \Lambda) = c < 1$$

then we can simply take $\rho = c$. From equation (8)

$$\| \nabla u_i^n \|^2_{L^2(\Omega_i)} = \int_{\Gamma_i} \frac{\partial u_i^n}{\partial n_i} u_i^n = \sum_\eta \lambda(\eta) \mid A_i^n(\eta) \mid^2 (e^{2\lambda(\eta)\delta} - e^{-4\lambda(\eta)l_i} e^{-2\lambda(\eta)\delta})$$

and by Poincaré inequality we have the estimate of the convergence rate in H^1 norm.

If there is no overlap i.e. $\delta = 0$, and $\lim_{|\eta|\to\infty} \left|\frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)}\right| \to 1$ then

$$\lim_{|\eta|\to\infty} \; r_G(\eta, \delta, l_1, l_2, \Lambda_1, \Lambda_2) \to 1 \quad (e.g.\ \Lambda_i = constant > 0)$$

But we have $u_i^n \in H^1(\Omega_i)$,

$$\frac{\partial u_i^n}{\partial n_i} \in H^{-\frac{1}{2}}(\Gamma_i), \qquad u_i^n \in H^{\frac{1}{2}}(\Gamma_i) \qquad \Rightarrow \qquad \int_{\Gamma_i} \frac{\partial u_i^n}{\partial n_i} u_i^n < \infty \qquad n = 0, 1, ...$$

or by (7)

$$\| \nabla u_i^n \|^2_{L^2(\Omega_i)} = \sum_\eta \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \infty$$

Now $\forall \epsilon > 0$, we can have a positive integer M such that

$$\sum_{|\eta|\geq M} \lambda(\eta) \mid A_i^0(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \frac{\epsilon}{2}$$

We can find N large enough such that for $n > N$,

$$\sum_{|\eta|< M} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \frac{\epsilon}{2}$$

since (12) is true. Then

$$\begin{aligned}
& \| \nabla u_i^n \|^2_{L^2(\Omega_i)} = \sum_\eta \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) \\
= \; & \sum_{|\eta|<M} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) + \sum_{|\eta|\geq M} \lambda(\eta) \mid A_i^n(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) \\
\leq \; & \frac{\epsilon}{2} + \sum_{|\eta|\geq M} \lambda(\eta) \mid A_i^0(\eta) \mid^2 (1 - e^{-4\lambda(\eta)l_i}) < \epsilon
\end{aligned}$$

Again by Poincaré inequality this shows that the domain decomposition algorithm is convergent even with no overlap. From the trace theorem or from the expression

$$\| u_i^n \|^2_{H^{1/2}(\Gamma_i)} = \sum_\eta (1+|\eta|)|\hat{u}_i^n(0,\eta)|^2$$

$$= \sum_\eta (1+|\eta|)|A_i^n|^2(1-e^{-2\lambda(\eta)l_i})^2 \le C\| \nabla u_i^n \|^2_{L^2(\Omega_i)}$$

We can get the convergence of $u_i^n|_{\Gamma_i} \to 0$ in $H^{1/2}(\Gamma_i)$. That completes the proof. $\square$

We can actually do the similar analysis (or just let $\Lambda = \infty$, $\Lambda = 0$) to get the convergence rate $r_D(\eta)$ for passing Dirichlet data, (classical SAM), and $r_N(\eta)$ for passing Neumann data at the artificial boundaries.

$$r_D(\eta) \stackrel{def}{=} \sqrt{\frac{e^{-\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_1}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}}} \times \sqrt{\frac{e^{-\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} - e^{-2\lambda(\eta)l_2}e^{-\lambda(\eta)\delta}}} \tag{13}$$

$$r_N(\eta) \stackrel{def}{=} \sqrt{\frac{e^{-\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_1}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_1}e^{-\lambda(\eta)\delta}}} \times \sqrt{\frac{e^{-\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_2}e^{\lambda(\eta)\delta}}{e^{\lambda(\eta)\delta} + e^{-2\lambda(\eta)l_2}e^{-\lambda(\eta)\delta}}} \tag{14}$$

It is easy to see that, if there is overlap, i.e. if $\delta > 0$ then

$$0 < r_D(\eta) < r_N(\eta) < 1, \qquad and \qquad \delta \to 0,\ r_N(\eta) \to 1,\ r_D(\eta) \to 1,\ \forall \eta$$

If we change the Dirichlet data to Neumann data at the boundary $\partial\Omega$ and do exactly the same calculation, then the above two expressions for r_D and r_N exchange. (Though in any case without overlap, these two algorithms alone do not have contraction for the error and thus do not converge.) But expression (11) for r_G remains the same. It is also not hard to see that the convergence rate $r_G(\eta)$ for the generalized SAM is a weighted form of the classical SAM $r_D(\eta)$.

Corollary 3.2. *For the generalized SAM (2) - (3), if the Dirichlet data at $\partial\Omega$ is changed to Neumann data, then Theorem 3.1 is still true.*

If we do some asymptotic analysis for in the interesting case where $l_1 \approx l_2 \gg \delta$, then

$$\begin{aligned} r_G(\eta) &\approx \left|\frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)}\right| e^{-2\lambda(\eta)\delta} \\ r_D(\eta) &\approx e^{-2\lambda(\eta)\delta} \\ r_N(\eta) &\approx e^{-2\lambda(\eta)\delta} \end{aligned} \tag{15}$$

Here we see that with small overlap the classical SAM only have a good convergence property on the highly oscillatory part in the solution, which corresponds to the well-known facts that iterative methods for elliptic PDE smooth out the error very quickly.

For the generalized SAM we can also have a good control of the smooth part or any range of modes in the error by an appropriate choice of operator Λ to approximate the DtN operator in that range.

DEFINITION 2. *Two positive operators Λ_1, Λ_2 are said to be equivalent in some Hilbert space H if*

$$\exists\ 0 < C_1 < C_2 \quad \textit{such that} \quad \forall u \in H \ \ 0 \leq C_1(\Lambda_1 u, u) \leq (\Lambda_2 u, u) \leq C_2(\Lambda_1 u, u)$$

where $(\cdot,\cdot)$ is the inner product in the Hilbert space.

If the positive operator in the generalized SAM is equivalent to the DtN operator, then the contraction rate due to the exchange of the mixed data is strictly less than 1. From the asymptotic expression (15) and the uniform convergence with overlap, we can easily see the following corollary.

COROLLARY 3.3. *For the generalized SAM (2) - (3), if the operators Λ_i is equivalent to the DtN operator in $H^{1/2}(\partial\Omega_i)$, then we have uniform convergence, i.e. $0 < r_G(\eta, \delta, l_1, l_2, \Lambda_1, \Lambda_2) < \rho < 1, \forall\eta, \forall\delta \geq 0$.*

Model Problem 2:

$$\begin{cases} -\triangle u = f & \textit{in } \Omega \\ \qquad u = g & \textit{on } \partial\Omega \end{cases}$$

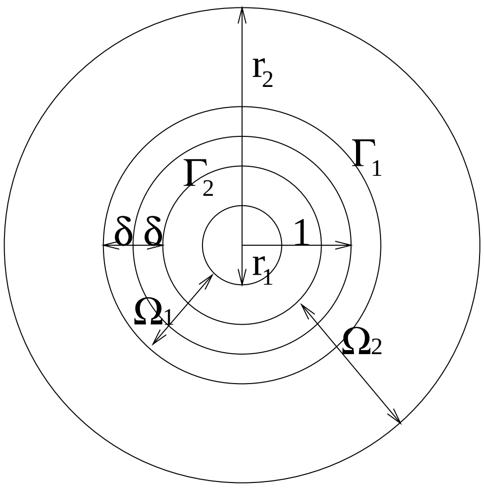


FIG. 3. *Circular domain with one subdomain possible inside the other*

$\Omega = \{(x,y) | r_1 \le \sqrt{x^2+y^2} \le r_2\}$
$\Omega_1 = \{(x,y) | r_1 \le \sqrt{x^2+y^2} \le 1+\delta\}$, $\Omega_2 = \{(x,y) | 1-\delta \le \sqrt{x^2+y^2} \le r_2\}$
$0 \le r_1 \le 1-\delta \le 1 \le 1+\delta \le r_2 < \infty$
$\Gamma_1 = \{(x,y) | \sqrt{x^2+y^2} = 1+\delta\}$, $\quad \Gamma_2 = \{(x,y) | \sqrt{x^2+y^2} = 1-\delta\}$

COROLLARY 3.4. *For model problem 2 Theorem3.1 holds for the additive generalized Schwarz alternating method, (2) - (3).*

Remark: If $0 < r_1 < r_2 < \infty$ it is easy to see from figure 3 that model problem 2 can be reduced to model problem 1 by an analytical transformation. However in the case $r_1 = 0$, i.e. $\Omega_1 \cap \partial\Omega = 0$ which is a very typical case in two or multi-subdomain situations, we can not use the standard Poincaré inequality as in the proof of Theorem3.1 to bound the H^1 norm by the H^1 seminorm. We are going to use the following lemma which is a direct extension of the lemma2 in [15] and proof can be found there.

LEMMA 3.5. *Let Ω be a bounded, open, smooth domain in R^N and γ_0 be an open(relative to $\partial\Omega$) subset of $\partial\Omega$. If $\Lambda: H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is a bounded positive operator, then there is a positive constant C such that*

$$\| u \|_{L_2(\Omega)} \le C \left(\| \nabla u \|_{L_2(\Omega)} + \| \frac{\partial u}{\partial n} + \Lambda u \|_{H^{-\frac{1}{2}}(\gamma_0)} \right) \tag{16}$$

for all $u \in H^1(\Omega)$ satisfying: $\triangle u = 0$ in Ω weakly, where n denotes the outward unit normal to $\partial\Omega$.

If $r_1 = 0$, i.e. in Ω_1 we do not see the boundary information, we need to use lemma 3.5 in Ω_1.

$$\begin{aligned}
\| u_1 \|_{L_2(\Omega_1)} &\le C \left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| \frac{\partial u_1}{\partial n_1} + \Lambda u_1 \|_{H^{-\frac{1}{2}}(\Gamma_1)} \right) \\
&\le C \left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| \frac{\partial u_2}{\partial n_2} \|_{H^{-\frac{1}{2}}(\Gamma_1)} + \| \Lambda u_2 \|_{H^{-\frac{1}{2}}(\Gamma_1)} \right) \\
&\le C \left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| u_2 \|_{H^1(\Omega_2)} \right) \\
&\le C \left(\| \nabla u_1 \|_{L_2(\Omega_1)} + \| \nabla u_2 \|_{L_2(\Omega_2)} \right)
\end{aligned}$$

where C denote some generic constant.

The proof of the corollary is very similar to that of model problem1 by using the polar coordinates $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$ and do Fourier transform in θ. The detail can be found in [24].

Using exactly the same argument except a index difference, we can show the same claim for the multiplicative version of the generalized SAM. In the case of two subdomains the convergence rate of the additive version is about the square root of that of the multiplicative version. If there are more than two subdomains the difference between the two versions depends on the number of colors we need to color the subdomians. (See [21])

COROLLARY 3.6. *For model problem 1 (or 2), Theorem 3.1 is true for the generalized multiplicative Schwarz alternating method, (4) - (5).*

Remark: If we look at (9), (10), each expression is not necessary less than 1 when the relative size of each subdomain is arbitrary, i.e. each one sided exchange of data at the artificial boundary may not be contractive. But a complete iteration will balance this effect. As a whole the generalized SAM is convergent even without overlap which agrees with the general energy estimate by P.L.Lions in [15]. In many applications, we have a comparable size of subdomains which are much larger than the size of overlap. Then each iteration will contribute about the same to the convergence.

4. Extension to General Equations in General Two Subdomain case. Now we replace the simple Laplace operator by the following elliptic operator,

$$\mathcal{L}u(\boldsymbol{x}) = -\bigtriangledown\cdot(a(\boldsymbol{x})\bigtriangledown u(\boldsymbol{x})) + \vec{b}(\boldsymbol{x})\cdot\bigtriangledown u(\boldsymbol{x}) + c(\boldsymbol{x})u(\boldsymbol{x}) \tag{17}$$

where $\boldsymbol{x} \in R^N$, $a(\boldsymbol{x})$, $\vec{b}(\boldsymbol{x})$, $c(\boldsymbol{x})$ are smooth and $a(\boldsymbol{x}) > 0$, $c(\boldsymbol{x}) \geq 0$. As long as we have the coercivity assumption as in [15], i.e. $\exists\ c > 0$, such that $\forall u \in H_0^1(\Omega)$

$$\int_\Omega a(\boldsymbol{x})|\bigtriangledown u(\boldsymbol{x})|^2 + \vec{b}(\boldsymbol{x})\cdot\bigtriangledown u(\boldsymbol{x})u(\boldsymbol{x}) + c(\boldsymbol{x})u^2(\boldsymbol{x})d\boldsymbol{x} \geq c\int_\Omega |\bigtriangledown u(\boldsymbol{x})|^2 d\boldsymbol{x} \tag{18}$$

then if $u(\boldsymbol{x})$ satisfies the homogeneous equation, $\mathcal{L}u(\boldsymbol{x}) = 0$, by divergence theorem we have

$$\int_\Omega a(\boldsymbol{x})|\bigtriangledown u(\boldsymbol{x})|^2 + \vec{b}(\boldsymbol{x})\cdot\bigtriangledown u(\boldsymbol{x})u(\boldsymbol{x}) + c(\boldsymbol{x})u^2(\boldsymbol{x})d\boldsymbol{x} = \int_{\partial\Omega}\frac{\partial u(\boldsymbol{x})}{\partial n}a(\boldsymbol{x})u(\boldsymbol{x}) \tag{19}$$

Using Poincaré inequality we can bound the H^1 norm of the solution by the boundary data. Now the only difference is that we have two bounded positive weight functions $a(\boldsymbol{x})$ and $c(\boldsymbol{x})$. Since elliptic differential operators are spectral equivalent to the Laplace operator in finite domain (see [7]), we have the spectral equivalence for the corresponding DtN operator.

For general geometries in the case of two subdomains, if the artificial boundaries are orthogonal to the real boundary, then using a smooth transformation we can transform them to one of the geometries in the model problems. If the artificial boundaries are not orthogonal to the real boundary or there are corners in the artificial boundaries, the DtN operator maybe singular at some points. The convergence rate for generalized SAM may deteriorate, though convergence is still true as long as the DtN operator is a positive operator and, $\int_{\Gamma_i} \frac{\partial u_i^n}{\partial n_i} u_i^n < \infty$, which is true by the trace theorem for Lipschitz domain. This can be seen from either the proof of the model problem or the proof of P.L.Lions in [15] using energy estimate.

THEOREM 4.1. *The generalized additive or multiplicative SAM (2)-(5) ($\lambda_1 = \lambda_2 = 1$) for elliptic differential operator (17) in the case of two subdomains is convergent if*

$$\Lambda_i = \Lambda : \; H^{\frac{1}{2}}(\partial\Omega_i) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega_i) \qquad i = 1, 2$$

is a strictly positive operator. If there is a uniform overlap, i.e. $dis(\Gamma_1, \Gamma_2) = \delta > 0$ then there is a constant $0 < \rho < 1$ such that

$$\| u - u_i^n \|_{H^1} \le \rho^n \| u - u_i^0 \|_{H^1} \; \; in \; \Omega_i, \; i = 1, 2$$

where ρ depends on Λ, the size of Ω_i, the size of the overlap δ and the partial differential equation.

From the analysis, we see that generalized SAM converges for the following two reasons,
(i) If there is overlap, maximum principle is true, i.e. the same mechanism as in the classical SAM works.
(ii) The artificial boundary conditions introduce some cancellation, which gives convergence even without overlap.

In many interesting cases the subdomains are of a comparable size which are much larger than the overlap. Using asymptotic analysis it can be shown that the overlap will give a contraction rate of $e^{-c\delta}$ (see [14]), where δ is the size of the overlap. The constant c will depend on the differential operator and the relative size of the subdomains. If we do eigen decomposition for the elliptic operator, c will depend on the eigen-mode and is a increasing function of the eigenvalue. That's why the classical SAM usually has a slow convergence speed for the smooth part of the solution which corresponds to eigen modes of the smaller eigenvalues. Coupled with the second factor in the generalized SAM, we have a good control on the smooth part of the solution. The following is some formal analysis on these two factors using operator formulation. Denote

$$\Omega_{12} = \Omega_1 \cap \Omega 2 \quad \Omega_{11} = \Omega_1 \backslash \Omega_{12} \quad \Omega_{22} = \Omega_2 \backslash \Omega_{12}$$

Define the projection operator

$$P_i : \; H^1(\Omega_i) \; \rightarrow \; H^1(\Omega_{ii})$$

to be the restriction from Ω_i to Ω_{ii}. Denote $D_i(D_{ii})$ to be the DtN operator in $\Omega_i(\Omega_{ii})$ on Γ_i. Let R_i be the restriction operator from $H^1(\Omega_i)$ to $H^{1/2}(\Gamma_i)$. Define R_i^* to be the right inverse operator of R_i, i.e. $R_i R_i^* = I$

$$\forall g \in H^{1/2}(\Gamma_i), \qquad \mathcal{L}(R_i^* g) = 0, \qquad R_i^* g = g \; at \; \Gamma_i \qquad R_i^* g = 0 \; at \; \partial\Omega_i \setminus \Gamma_i$$

Denote R_{ii}, R^*_{ii} to be the corresponding operators in Ω_{ii}

After one cycle of iteration on the error function $e^n_i = u - u^n_i$ in Ω_i for the generalized additive SAM (2), (3), formally we have

$$e^n_i = R^*_i(\Lambda_i + D_i)^{-1}(\Lambda_i - D_{jj})R_{jj}P_jR^*_j(\Lambda_j + D_j)^{-1}(\Lambda_j - D_{ii})R_{ii}P_i(e^{n-2}_i) \tag{20}$$

$\{i, j\} = \{1, 2\},\ i \neq j$. Two extreme cases:

(i) In the case of classical SAM, (20) is reduced to

$$e^n_i = R^*_i R_{jj} P_j R^*_j R_{ii} P_i(e^{n-2}_i)$$

(ii) In the case of no overlap, (20) is reduced to

$$e^n_i = R^*_i(\Lambda_i + D_i)^{-1}(\Lambda_i - D_j)(\Lambda_j + D_j)^{-1}(\Lambda_j - D_i)R_i(e^{n-2}_i)$$

The convergence of the generalized SAM can be regarded as the product of the two factors when the overlap is small. The first factor of convergence is the projection operator. The second factor is $(\Lambda_i + D_i)^{-1}(\Lambda_i - D_{jj})$, which is the result of the exchange of convex combination of u and $\frac{\partial u}{\partial n}$ at the artificial boundary.

$$\begin{array}{ll} u^n_1 = (\Lambda_1 + D_1)^{-1}(\Lambda_1 - D_{22})u^{n-1}_2 & \quad at\ \Gamma_1 \\ u^n_2 = (\Lambda_2 + D_2)^{-1}(\Lambda_2 - D_{11})u^{n-1}_1 & \quad at\ \Gamma_2 \end{array} \tag{21}$$

and

$$\begin{array}{rcl} (\Lambda_1 + D_1)^{-1}(\Lambda_1 - D_{22}) & = & I - (\Lambda_1 + D_1)^{-1}(D_1 + D_{22}) \\ (\Lambda_2 + D_2)^{-1}(\Lambda_2 - D_{11}) & = & I - (\Lambda_2 + D_2)^{-1}(D_2 + D_{11}) \end{array} \tag{22}$$

where Λ_i, D_i, D_{ii} are all positive operators. The contraction rate for the second factor is determined by the spectral behavior of $\Lambda_i^{-1}D_j$ and $\Lambda_i^{-1}D_{jj}$. Each individual expression in (22) dose not necessary have a norm less than 1. It depends on the relation between Λ_i and D_i. But the product of the two is contractive. Some discussions on the eigen decompositions of the interface operators after discretization can be found in [2].

Remark: If Λ_i changes in each iteration, then we can damp out errors in different range of frequencies more efficiently. This is the motivation for the accelerated version of the generalized SAM. Some numerical results are shown in section6. The optimal choice of the sequence Λ_i is under study.

Remark: Now let's apply the above analysis briefly to the interface problem. Consider the elliptic PDE,

$$
\begin{array}{ll}
\nabla \cdot (\beta(\boldsymbol{x}) \nabla u(\boldsymbol{x})) + \kappa(\boldsymbol{x}) u(\boldsymbol{x}) = f(\boldsymbol{x}) & \\
\beta(\boldsymbol{x}) > 0, \kappa(\boldsymbol{x}) > 0, \qquad \boldsymbol{x} \in \Omega \subset R^N, \; N = 1, 2, 3 &
\end{array}
\tag{23}
$$

in a domain Ω. In Ω, suppose there is an interface of co-dimension one across which the solution u and/or some of its derivatives are discontinuous. (see figure 4) The discontinuity may be caused by some singular source in f at the interface such as surface tension and/or may be caused by discontinuities in the coefficients $\beta(\boldsymbol{x}), \kappa(\boldsymbol{x})$ due to different material properties.

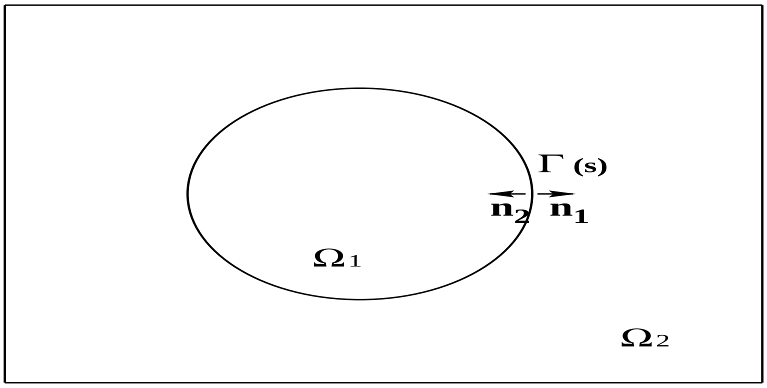


FIG. 4. *Interface problem*

From the equation or some physical arguments the interface conditions can be described by some jump conditions and flux conditions across the interface in the following form,

$$
\left\{ \begin{array}{l}
[u]|_{\Gamma(s)} = u_1|_{\Gamma(s)} - u_2|_{\Gamma(s)} = h(s) \\
\left[\beta u_n\right]|_{\Gamma(s)} = \beta_1 \frac{\partial u_1}{\partial n}|_{\Gamma(s)} - \beta_2 \frac{\partial u_2}{\partial n}|_{\Gamma(s)} = k(s)
\end{array} \right.
\tag{24}
$$

where $(X(s), Y(s))$ is a parameterization of the interface Γ, n is the outward normal of Γ. (see figure 4) The PDE (23) in each subdomain and the interface conditions (24) will uniquely determine the solution u. This is a quite challenging problem. In standard numerical schemes some continuity in the solution and its derivatives are very needed to achieve the accuracy or convergence estimates. Here the worst happens. There are some methods developed to handle this situation. The boundary integral methods in [17],[18] using potential theory, require β to be piecewise constant. The immersed boundary method in [19] using a numerical delta function along the interface, can only handle the discontinuity in the normal derivative and may smear the sharp interface. Recently R. LeVeque and Z. Li has developed a quite versatile immersed interface method in [11] using a modified finite difference scheme at the interface according to the jump conditions. It has been applied very successfully to many applications in [12],[8]. Here we will modify the multiplicative version of the generalized SAM to the interface problem.

Set u_i^0 initially $i = 1, 2$, Λ^1, Λ^2 are some positive operators. Define a full cycle of iteration to be the following two consecutive steps of generalized SAM.
First step.

$$\begin{cases} -\nabla \cdot (\beta_1(x) \nabla u_1^{2n+1}) = f & in\ \Omega_1 \\ u_1^{2n+1} = g & at\ \partial\Omega \cap \partial\Omega_1 \\ \Lambda^1 u_1^{2n+1} + \beta_1 \frac{\partial u_1^{2n+1}}{\partial n_1} = \Lambda^1 u_2^{2n} + \beta_2 \frac{\partial u_2^{2n}}{\partial n_1} + \Lambda^1 h + k\ at\ \Gamma \end{cases}$$

$$\begin{cases} -\nabla \cdot (\beta_2(x) \nabla u_2^{2n+2}) = f & in\ \Omega_2 \\ u_2^{2n+2} = g & at\ \partial\Omega \cap \partial\Omega_2 \\ \Lambda^1 u_2^{2n+2} + \beta_2 \frac{\partial u_2^{2n+2}}{\partial n_2} = \Lambda^1 u_1^{2n+1} + \beta_1 \frac{\partial u_1^{2n+1}}{\partial n_2} - \Lambda^1 h + k\ at\ \Gamma \end{cases}$$

Second step is almost the same as the first step but replace Λ^1 by Λ^2. The reason why we need two Λ's is obvious from the following lemma.

LEMMA 4.2. *The jump condition(24) is equivalent to the following jump conditions*

$$\begin{cases} \Lambda_1[u] + [\beta u_n] = \Lambda_1 h(s) + k(s), & at\ \Gamma \\ \Lambda_2[u] + [\beta u_n] = \Lambda_2 h(s) + k(s), & at\ \Gamma \end{cases}$$

if Λ_i's are strictly positive and different. $i = 1, 2$.

It is easy to see that the error functions $e_i^n = u - u_i^n$ in $\Omega_i, i = 1, 2$ satisfy the following homogeneous equations and interface conditions.

$$\begin{cases} -\nabla \cdot (\beta_1(x) \nabla e_1^{2n+1}) = 0 & in\ \Omega_1 \\ e_1^{2n+1} = 0 & at\ \partial\Omega \cap \partial\Omega_1 \\ \Lambda^m e_1^{2n+1} + \beta_1 \frac{\partial e_1^{2n+1}}{\partial n_1} = \Lambda^m e_2^{2n} + \beta_2 \frac{\partial e_2^{2n}}{\partial n_1} & at\ \Gamma \end{cases}$$

$$\begin{cases} -\nabla \cdot (\beta_2(x) \nabla e_2^{2n+2}) = 0 & in\ \Omega_2 \\ e_2^{2n+2} = 0 & at\ \partial\Omega \cap \partial\Omega_2 \\ \Lambda^m e_2^{2n+2} + \beta_2 \frac{\partial e_2^{2n+2}}{\partial n_2} = \Lambda^m e_1^{2n+1} + \beta_1 \frac{\partial e_1^{2n+1}}{\partial n_2} & at\ \Gamma \end{cases}$$

$n = 0, 1, 2, \ldots,\ m = 1, 2$. So formally the convergence rate of this procedure for the interface problem can be seen from the error reduction in each iteration.

$$e_i^n = R_i^{-1}(\Lambda^m + \beta_i D_i)^{-1}(\Lambda^m - \beta_j D_j)(\Lambda^m + \beta_j D_j)^{-1}(\Lambda^m - \beta_i D_i) R_i(e_i^{n-2})$$

where $i, j, m = 1, 2$ and the operators are defined as in (20). Since the error in each subdomain satisfies a well behaved PDE and boundary conditions, the high frequency components in the error is small. The smooth components in the error is damped quickly by the mixed boundary conditions at the interfaces. We could also use the accelerated version of the generalized SAM to minimize the effect of the jump size and improve the convergence speed. Cartesian grids can be used. The details will be discussed in later papers.

5. Local Operator Implementation of the Generalized SAM. We see from previous sections that the convergence rate of the generalized SAM depends on the size of overlap, the PDE and the choice of the positive operator at the artificial boundary. Theoretically the best choice is the DtN operator which is a global operator in most cases and is not practical in numerical implementation. Motivated by the absorbing type boundary condition proposed by Engquist and Majda in [6], we will use a local operator to approximate the DtN operator by truncating an asymptotic expansion of a global pseudo differential operator. This operator will contain derivatives tangential to the artificial boundary. We start with a model problem

$$\begin{cases} - \triangle u + u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{25}$$

where $\Omega, \Omega_1, \Omega_2$ and the generalized SAM are the same as in model problem 1 in section3. From the analysis in model problem 1 the convergence rate of the generalized SAM in each frequency mode is

$$r_G \approx \left| \frac{\lambda(\eta) - \Lambda(\eta)}{\lambda(\eta) + \Lambda(\eta)} \right| e^{-2\lambda(\eta)\delta}$$

if $l_1 \approx l_2 \gg \delta$. Now $\lambda(\eta) = \sqrt{1 + \eta^2}$. When solving elliptic equations the smooth part of the solution converges very slowly. We expand $\lambda(\eta)$ at 0.

$$\lambda(\eta) = \sqrt{1 + \eta^2} = 1 + \frac{1}{2}\eta^2 + O(\eta^4)$$

Using first order Taylor expansion we get the first order local approximation.

$$\Lambda(\eta) = 1 \;\Rightarrow\; \Lambda u = u \tag{26}$$

Using second order Taylor expansion we get the second order local approximation assuming the boundary and the solution make $\frac{\partial^2 u}{\partial \tau^2}$ meaningful.

$$\Lambda(\eta) = 1 + \frac{1}{2}\eta^2 \;\Rightarrow\; \Lambda u = u - \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} \tag{27}$$

where τ is the tangential direction.

For the discretized problem suppose that the grid size is $h \ll 1$ and overlap is one grid size, i.e. $\delta = h$, then the highest frequency mode is $\eta_{max} = \dfrac{1}{h}$.

$$r_G \approx \max_{0\le\eta\le\eta_{max}} \left| \frac{\sqrt{1+\eta^2} - \Lambda(\eta)}{\sqrt{1+\eta^2} + \Lambda(\eta)} \right| e^{-2h\sqrt{1+\eta^2}}$$

Differentiate this expression and find the equation for the critical value η_c is

$$\eta\Lambda(\eta) - (1+\eta^2)\Lambda'(\eta) = \eta h(1 + \eta^2 - \Lambda^2(\eta)) \tag{28}$$

For the first order approximation (34), we have $\eta_c = \sqrt{\dfrac{1}{h}}$ and hence

$$r_G \approx \frac{\sqrt{1+h^{-1}} - 1}{\sqrt{1+h^{-1}} + 1} e^{-2h\sqrt{1+h^{-1}}} \approx 1 - 4h^{\frac{1}{2}}$$

For the second order approximation (35) we have $\eta_c = \sqrt{\dfrac{2}{h}}$ and hence

$$r_G \approx \frac{\sqrt{1+h^{-1}} - (1+h^{-1})}{\sqrt{1+h^{-1}} + (1+h^{-1})} e^{-2h\sqrt{1+2h^{-1}}} \approx 1 - 8h^{\frac{1}{2}}$$

These estimates are both conservative. In real numerical computations (see section6) the convergence is usually faster, since the solutions to the elliptic PDEs are smooth, i.e. the high frequency components in the error are small. That is why we choose Taylor expansion near 0. If the constant function is in the null space of the differential operator then boundary information is needed. In multi-subdomain case, mutli-scale domain decomposition may be used to accelerate the propagation of the boundary information through the subdomains.

The convergence rate for classical SAM is,

$$r_D \approx \max_{0\le\eta\le\eta_{max}} e^{-2\lambda(\eta)\delta} \approx e^{-h} \approx 1 - h$$

We notice that the worst convergence occurs near $\eta = 0$, which is a significant part of the solution. That is why the classical SAM can be sensitive to the size of overlap.

Although we can improve the convergence using generalized SAM, it is clear that the convergence rates for the first and second order schemes still depend on the size of overlap. Now let us address the question of optimality of our numerical schemes by some careful asymptotic analysis. In the following presentation c always means some constant which is independent of mesh size h and they are not necessary the same.

(i) First order scheme:

Let $\Lambda(\eta) = ch^{\alpha}$ and plug it into the equation(28) for the critical point.

$$\eta_c^2 = c^2h^{2\alpha} + ch^{\alpha-1} - 1 \qquad or \qquad \eta_c = 0 \tag{29}$$

If (a) $\alpha \geq 1$ then

$$r_G(0) = \frac{1-ch^{\alpha}}{1+ch^{\alpha}}e^{-2h} \approx (1-ch^{\alpha})(1-2h) \approx 1-2h$$

(b)$0 \leq \alpha < 1$ then

$$r_G(0) \approx (1-ch^{\alpha})(1-2h) \approx 1-ch^{\alpha}$$

and at $\eta_c^2 \approx ch^{\alpha-1}$

$$r_G(\eta_c) = \frac{\sqrt{1+ch^{\alpha-1}}-ch^{\alpha}}{\sqrt{1+ch^{\alpha-1}}+ch^{\alpha}}e^{-2h\sqrt{1+ch^{\alpha-1}}} \approx (1-ch^{\frac{1+\alpha}{2}})(1-ch^{\frac{1-\alpha}{2}}) \approx 1-ch^{\frac{1-\alpha}{2}}$$

So the best choice is $\alpha = \dfrac{1}{3} \ \Rightarrow\ r_G = 1-ch^{\frac{1}{3}}$

(c) $-1 \leq \alpha < 0$ then

$$r_G(0) \approx (1-ch^{-\alpha})(1-2h) \approx 1-ch^{-\alpha}$$

at $\eta_c^2 \approx ch^{\alpha-1}$

$$r_G(\eta_c) = \frac{\sqrt{1+ch^{\alpha-1}}-ch^{\alpha}}{\sqrt{1+ch^{\alpha-1}}+ch^{\alpha}}e^{-2h\sqrt{1+ch^{\alpha-1}}} \approx (1-ch^{\frac{1+\alpha}{2}})(1-ch^{\frac{1-\alpha}{2}}) \approx 1-ch^{\frac{1+\alpha}{2}}$$

Hence the best choice is $\alpha = -\dfrac{1}{3} \ \Rightarrow\ r_G = 1-ch^{\frac{1}{3}}$

(d) $\alpha < -1$ then

$$r_G(0) \approx (1-ch^{-\alpha})(1-2h) \approx 1-2h$$

For the first order scheme the optimal convergence rate is $1-ch^{\frac{1}{3}}$ when we take $\Lambda = O(h^{\frac{1}{3}})$ or $\Lambda = O(h^{-\frac{1}{3}})$. We prefer to take $\Lambda = O(h^{-\frac{1}{3}})$ since the critical value in this case is at $\eta_c = O(h^{-\frac{2}{3}})$ which is much larger than the other choice.

(ii) Second order scheme:

Let $\Lambda(\eta) = 1 + ch^{\alpha}\eta^2$. Again substitute it into equation(28) for the critical value.

$$c^2h^{2\alpha+1}\eta^4 + (2ch^{\alpha+1} - h - ch^{\alpha})\eta^2 + (1 - 2ch^{\alpha}) = 0 \qquad or \qquad \eta = 0 \tag{30}$$

so

$$\eta_c^2 = \frac{(h + ch^{\alpha} - 2ch^{\alpha+1}) \pm \sqrt{(h + ch^{\alpha} - 2ch^{\alpha+1})^2 - 4c^2h^{2\alpha+1}(1 - 2ch^{\alpha})}}{2c^2h^{2\alpha+1}} \qquad or\ \eta_c = 0$$

and at $\eta = 0$, $r_G(0) \approx 0$. We now divide the study into four cases and do some asymptotic analysis again.

(a) If $\alpha \leq -1$,

$$\eta_c^2 = \frac{(h + ch^{\alpha} - 2ch^{\alpha+1}) \pm \sqrt{(h + ch^{\alpha} - 2ch^{\alpha+1})^2 - 4c^2h^{2\alpha+1}(1 - 2ch^{\alpha})}}{2c^2h^{2\alpha+1}} \approx ch^{-\frac{\alpha+1}{2}}$$

and

$$r_G \approx \left| \frac{\sqrt{1 + ch^{-\frac{\alpha+1}{2}}} - (1 + ch^{\frac{\alpha-1}{2}})}{\sqrt{1 + ch^{-\frac{\alpha+1}{2}}} + (1 + ch^{\frac{\alpha-1}{2}})} \right| e^{-2h\sqrt{1+ch^{-\frac{\alpha+1}{2}}}} \geq 1 - ch$$

(b) If $-1 < \alpha \leq 0$,

$$\eta_c^2 \approx ch^{-(\alpha+1)}$$

and $\qquad r_G \approx 1 - ch^{\frac{1-\alpha}{2}} \geq 1 - ch^{\frac{1}{2}}$

(c) If $0 < \alpha \leq 1$,

$$\eta_c^2 \approx ch^{-(\alpha+1)} \qquad or \qquad \eta_c^2 \approx ch^{-\alpha}$$

and $\qquad r_G \approx max(1 - ch^{\frac{1-\alpha}{2}},\ (1 - ch^{\frac{\alpha}{2}})(1 - ch^{1-\frac{\alpha}{2}}))$

So the best choice is when $\frac{\alpha}{2} = \frac{1-\alpha}{2}$, i.e. $\alpha = \frac{1}{2}$ and $r_G \approx 1 - ch^{\frac{1}{4}}$.

(d) If $\alpha > 1$,

$$\eta_c^2 \approx ch^{-2\alpha} \qquad or \qquad \eta_c^2 \approx ch^{-1}$$

and $\qquad r_G \approx 1 - ch^{\frac{1}{2}}$

So the optimal choice for the second order scheme is

$$\Lambda u = u - ch^{\frac{1}{2}}\frac{\partial^2 u}{\partial \tau^2} \tag{31}$$

(τ is the tangential direction at the interface) for the generalized SAM and the convergence rate is $1 - ch^{\frac{1}{4}}$.

In the variable coefficients case, we can (i) either use our first or second order approximation to the homogenized (averaged) equation (ii) or just use the local approximation at the artificial boundary. If the coefficients are highly oscillatory, the first choice would be better since the large scale behavior of the solution is more close to the homogenized equation. The highly oscillatory part of the solution can be resolved by the solver in each subdomain. If the coefficients are smooth, we suggest to use the second choice since we are using local operators to approximate the global one in anyway.

Remark: Since we do not have the exact data at the artificial boundary it does not make sense to solve the problem in each subdomain exactly. Instead we can use some efficient iterative scheme to some tolerance. We only need that the non-exact solver is accurate enough, i.e. the error e satisfies $P(D)e = \tilde{f}$ and $\tilde{f}$ is small enough, such that for e maximum principle and positivity of the DtN operator still hold. A study on the dynamical control of the tolerance can be found in [5].

6. Numerical Results.

In this section we shall present numerical calculations to quantitatively illustrate some of the analytical results from previous sections.

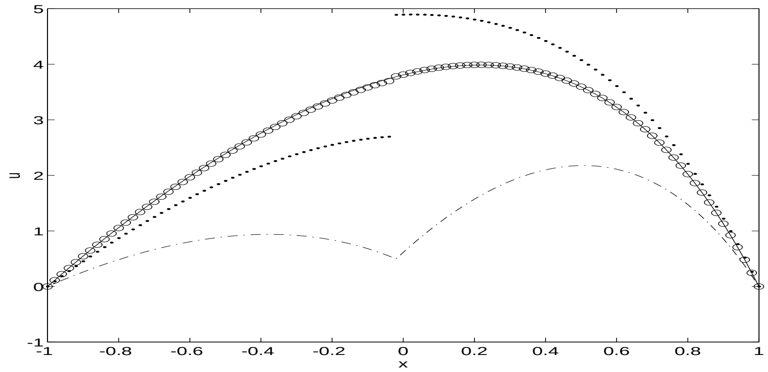
Example 1: First we consider the 1-D two points boundary value problem,

$$\begin{array}{ll} -u_{xx}(x) + u(x) = 10e^x & in\ \Omega = [-1, 1] \\ u(-1) = 0,\ u(1) = 0 & \\ \Omega_1 = [-1, 0.02],\ \Omega_2 = [-0.02, 1] & (i.e.\ the\ overlap\ is\ 0.04) \end{array}$$

We used the Dirichlet, Neumann and mixed type($\Lambda_i = 1$) boundary condition at the artificial boundary. Central difference is used to discretize the differential equation with grid size $h = 0.02$. We use exact solver in each subdomain in our multiplicative domain decomposition method. From figure 5 we see that after only two iterations, the solution to our mixed boundary condition is not only much closer to the true solution but is also smoother since mixed data are exchanged. Also from the L_2 error plot figure 6, we see that both the error and the convergence rate for the mixed boundary condition are better than the other two methods which agrees with our analysis. We test the optimality for different choice of the $\Lambda_i = \Lambda$ by letting $\Lambda = \frac{\alpha}{1-\alpha}, 0 < \alpha < 1$. Figure 7 shows that our choice of parameter $\Lambda = 1$ in the mixed boundary condition which cancels the dominant term in the error is almost the optimal one for our 1-D problem.

Example 2:

$$\begin{array}{ll} -\triangle u(x, y) + u(x, y) = 10e^x(1-y)y & in\ \Omega = [-1, 1] \times [0, 1] \\ u = 0 & on\ \partial\Omega \end{array}$$



exact solution -, solution with mixed condition o,
solution with Dirichlet condition ., solution with Neumann condition-.

FIG. 5. *Solution plot for example 1*

$\Omega_1 = [-1, 0.02] \times [0,1],\ \Omega_2 = [-0.02, 1] \times [0,1]\ \Gamma_1 = \{0.02\} \times [0,1],\ \Gamma_2 = \{-0.02\} \times [0,1]$

Five points central difference scheme is used with grid size $h = 0.02$. Again exact solver in each subdomain and multiplicative version of the domain decomposition method is used. We use the following four different boundary conditions at the artificial boundaries $\Gamma_1,\ \Gamma_2$.

(1) Dirichlet boundary condition:

$$u_i^{(n+1)} = u_j^{(n)} \qquad on\ \Gamma_i \tag{32}$$

(2)Neumann boundary condition:

$$(u_i^{(n+1)})_x = (u_j^{(n)})_x \qquad on\ \Gamma_i \tag{33}$$

(3) first order mixed boundary condition:

$$(u_i^{(n+1)})_x \pm u_i^{(n+1)} = (u_j^{(n)})_x \pm u_j^{(n)} \qquad on\ \Gamma_i \tag{34}$$

(4) second order boundary condition:

$$(u_i^{(n+1)})_x \pm u_i^{(n+1)} \mp \frac{1}{2}(u_i^{(n+1)})_{yy} = (u_j^{(n)})_x \pm u_j^{(n)} \mp \frac{1}{2}(u_j^{(n)})_{yy} \quad on\ \Gamma_i \tag{35}$$

$\{i, j\} = \{1, 2\},\ i \neq j$

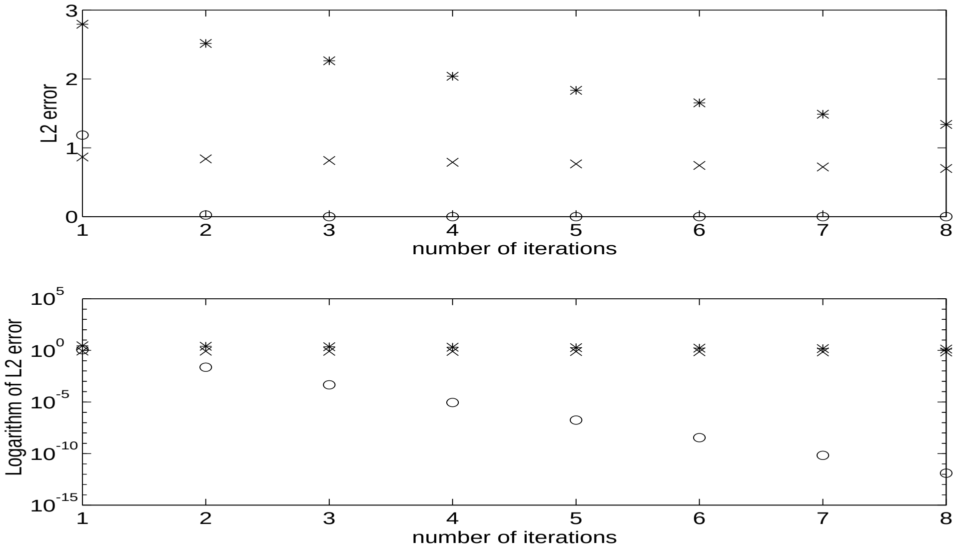
Figure 8 is the numerical results with different boundary conditions.

Example3:

$$\begin{aligned} -\Delta u(x,y) + u(x,y) = xy &\quad in\ \Omega = [-1,1] \times [0,1] \\ u = e^x y + xy &\quad on\ \partial\Omega \end{aligned}$$

The exact solution is $u(x,y) = e^x y + xy$.

$\Omega_1 = [-1, \delta] \times [0,1],\ \Omega_2 = [-\delta, 1] \times [0,1]\ \Gamma_1 = \{\delta\} \times [0,1],\ \Gamma_2 = \{-\delta\} \times [0,1]$



mixed o, Dirichlet *, Neumann x

FIG. 6. *Error plot for example 1*

In the computation we will change the size of overlap with mesh size h, i.e. $\delta = h$. We use SOR with tolerance 10^{-6} for the system in each subdomain discretized by central difference. We tested the classical SAM (32), first order (34) and second order (35) generalized SAM schemes in both additive and multiplicative version of domain decomposition method. Figure 9 shows the convergence of the multiplicative domain decomposition method in the L_2 norm as $h \to 0$. We see that the convergence of the generalized SAM is relative insensitive to the size of overlap. Figure 10 shows the convergence of the additive domain decomposition method.

Figure 11 shows that both the multiplicative version and the additive version of the generalized SAM converge without overlap but not in any geometric fashion. However

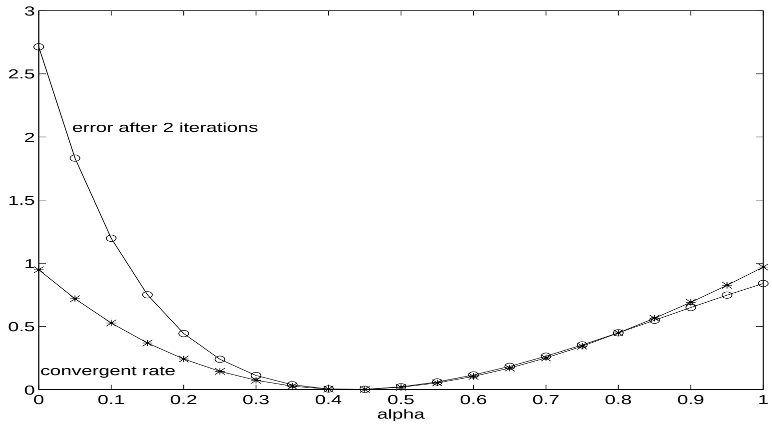
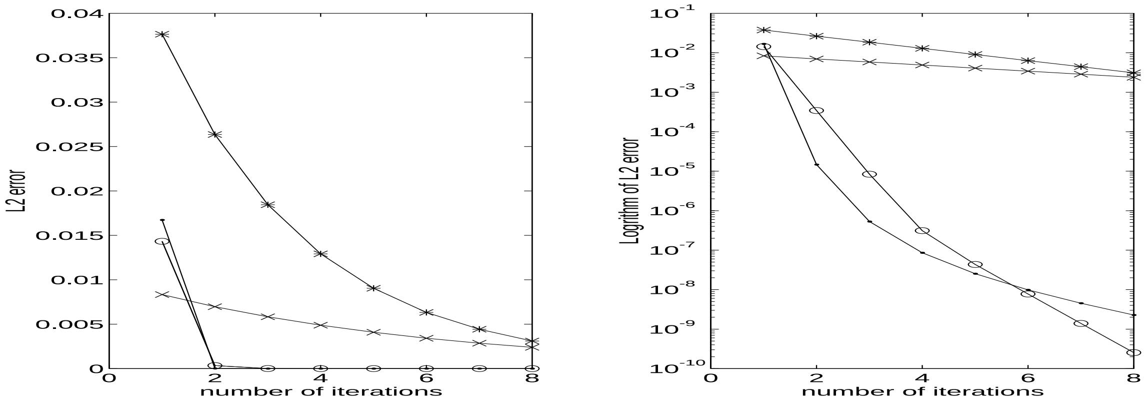
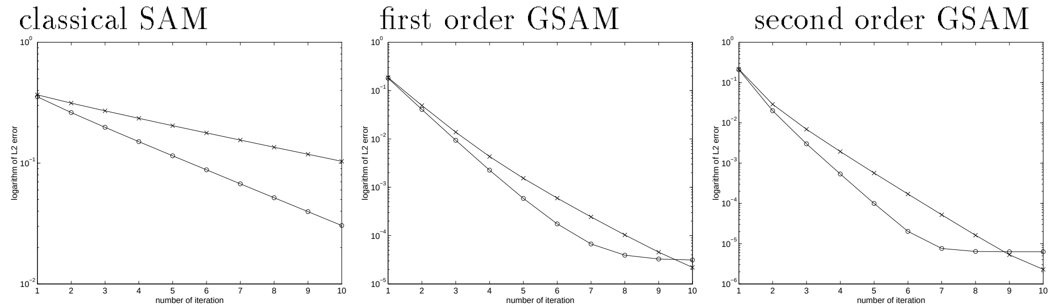


FIG. 7. *Errors and convergence rates with different* Λ.



second order ., first order o, Dirichlet *, Neumann x

FIG. 8. *Error plot for example 2*



'o' for $\delta = h = 0.02$, 'x' for $\delta = h = 0.01$

FIG. 9. *Multiplicative SAM*

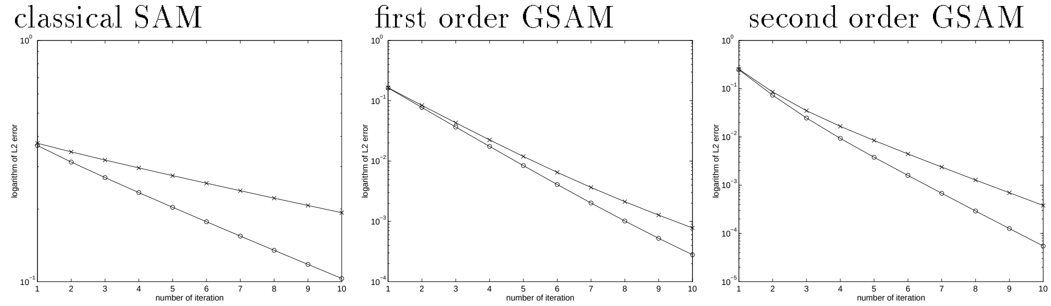
they can be dramatically improved by using accelerated version (see section4 Remark), i.e. using different Λ_i^n in each iteration. In this calculation, we use $\Lambda_i^n = 1 + n, n = 1, 2, \ldots, 20$ at nth iteration.

Example 4: We will test our accelerated generalized SAM i.e. $\Lambda_i^n = 1 + n, n = 1, 2, \ldots, 10$ at nth iteration, on a second order elliptic equation with variable coefficients.

$$
\begin{array}{ll}
-\nabla \cdot ((1 + 0.8 sin\pi xy)\nabla u(x, y)) = e^{x+y} & in\ \Omega = [-1, 1] \times [0, 1] \\
u = 0 & on\ \partial\Omega
\end{array}
$$

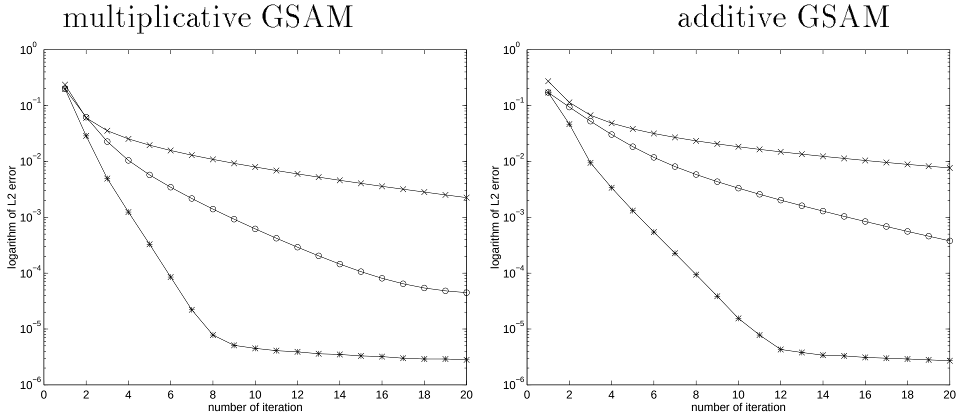
The whole domain is divided into two subdomains with different sizes.
$\Omega_1 = [-1, -0.7 + \delta] \times [0, 1],\ \Omega_2 = [-0.7 - \delta, 1] \times [0, 1]$
$\Gamma_1 = \{-0.7 + \delta\} \times [0, 1],\ \Gamma_2 = \{-0.7 - \delta\} \times [0, 1]$

We use grid size $h = 0.01$ and change the size of overlap. The same SOR method is used in each subdomain as in Example 3. From figure 12 we see that both versions of the accelerated generalized SAM converge very fast and is almost independent of the size of overlap. When the size of overlap is large enough then the convergence effect due to overlap takes over.



'o' for $\delta = h = 0.02$, 'x' for $\delta = h = 0.01$

FIG. 10. *Additive SAM*



'x' for 1st order, 'o' for 2nd order, '*' for the accelerated version, $h = 0.02$

FIG. 11. *Convergence of GSAM without overlap*

7. Conclusion. Due to the positivity of the Dirichlet to Neumann operator the generalized SAM can have improved convergence properties over the classical SAM. The generalized SAM for domain decomposition can converge without overlap. Since a local operator can not approximate a global operator uniformly in general, we can not expect a convergence rate which is independent of the overlapping size for the generalized SAM.

We will discuss the accelerated version of generalized SAM, its extension to multi-domain cases and the application to interface problems in some other papers.

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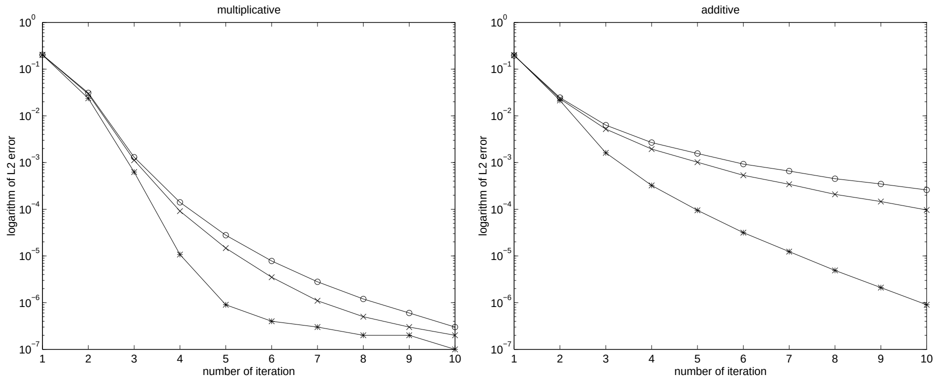
multiplicative additive



'o' for $\delta = 0$, 'x' for $\delta = h$, '*' for $\delta = 5h$, $h = 0.01$

FIG. 12. *Convergence of accelerated GSAM*

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