Updating an Abel-Gauss-Riemann Program

Versions: 1st UC Irvine 05/22/08, 2nd Istanbul 06/18/08

In "What Gauss Told Riemann About Abel's Theorem" (lecture at John Thompson's 70th Birthday) I cited Otto Neuenschwanden on 60-year-old Gauss in conversation with 20-year-old Rieman.

Their goal: Generalize two of Abel's famous results using Gauss' harmonic functions. Notation:

$$\boldsymbol{g}_{\text{H-M}} = (g_1, g_1^{-1}, g_2, g_2^{-1}) \in \text{Ni}(G, \mathbf{C}) \text{ for an H-M rep.}$$

Two understatements: Riemann went far; but his early death left an incomplete program.

- §I. More modular curve lessons
- §II. Modular curve-like spaces with A_n s replacing D_p s

§I. More modular curve lessons: Extending [Talk$_1$]

Nielsen class Reminder: Finite G and r conjugacy classes **C**: $\boldsymbol{g} = (g_1, \dots, g_r) \in \mathrm{Ni}(G, \mathbf{C})$ satisfies:

- (generation) $\langle \boldsymbol{g} \rangle = G$;
- (conjugacy classes – with multiplicity) $\boldsymbol{g} \in \mathbf{C}$; and
- (product-one – in the given order) $g_1 \dots g_r = 1$.

Dragging a dihedral function by its branch points gives: $\mathcal{H}(D_p, \mathbf{C}_{2^4}) \stackrel{\text{def}}{=} \{f \in \mathrm{Ni}(D_p, \mathbf{C}_{2^4})\}$ complex and an analytic map, $\Psi_{D_p, \mathbf{C}_{2^4}}$, to (j_w, j_z)-space with image $Y_0(p)$.

§I.A. Here are the cusps, at all levels

Chow Lemma: Extend $\Psi_{D_p,\mathbf{C}_{2^4}}$ to include cusps $\implies$ equations in (j_w, j_z) for $Y_0(p) \cup \text{cusps} \stackrel{\text{def}}{=} X_0(p)$.

Orbits means on $\mathrm{Ni}(D_p, \mathbf{C}_{2^4})^{\mathrm{abs}} \stackrel{\text{def}}{=} \mathrm{Ni}(D_p, \mathbf{C}_{2^4})/N_p$ with

$$N_p = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\}_{a \in (\mathbb{Z}/p)^*, b \in \mathbb{Z}/p}.$$

- Points on $X_0(p)$ over $j_z \neq \infty \leftrightarrow \mathcal{Q}''$ orbits.
- Cusps (over $j_z = \infty$) $\leftrightarrow \mathrm{Cu}_4 \stackrel{\text{def}}{=} \langle q_2, \mathcal{Q}'' \rangle$ orbits. Just two: $\mathrm{H\text{-}M}$ rep. $(g_1, g_1^{-1}, g_2, g_2^{-1}) = \boldsymbol{g}_{\mathrm{H\text{-}M}}$, in this case a p-cusp; $p | \mathrm{ord}(g_1^{-1} g_2))$;
  other by $(\boldsymbol{g}_{\mathrm{H\text{-}M}})$**sh** (g-$p'$ cusp; $p \not| |\langle g_2, g_2^{-1} \rangle|$ or $|\langle g_1, g_1^{-1} \rangle|$).

§I.B. Generalize T_p compositions $T_{p^{k+1}} = T_p \circ \cdots \circ T_p$: Use Schur-zassenhaus and Frattini Properties

For $X_0(p^{k+1})$: Exchange $D_{p^{k+1}}$ for D_p, $N_{p^{k+1}}$ for N_p.

Form modular curves $X_1(p^{k+1})$: Replace $\mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})/N_{p^{k+1}}$ by $\mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})/D_{p^{k+1}} = \mathrm{Ni}(\dots)^{\mathrm{in}}$.

Schur-Zassenhaus: Regard C_2 as a conjugacy class in each $D_{p^{k+1}}$.

Generation at level k (Frattini): $\boldsymbol{g} \in \mathbf{C}_{2^4} \cap D_{p^{k+1}} \implies \langle \boldsymbol{g} \rangle = D_{p^{k+1}}$ iff $\langle \boldsymbol{g} \mod p \rangle = D_p$. So, to list $\mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})$ track just product-one condition.

§I.C. (product-one) D_p-iterations

Abel Iteration: Functions $f_k : \mathbb{P}^1_w \to \mathbb{P}^1_z$ with branch locus $j' \in U_j$ modulo PGL_2 in $\mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}} \implies$ projective sequence of covers in $\mathrm{Ni}(D_{p^t}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$, $1 \le t \le k+1$.

Lemma: For ${}_0\boldsymbol{g} \in \mathrm{Ni}(D_p, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$ a b(ranch)c(yc)d(esc) realized by $f_0 : \mathbb{P}^1_w \to \mathbb{P}^1_z$, substitute another group G_t for $D_{p^{t+1}}$ with $\ker(G_t \to G_{t-1}) = \mathbb{Z}/p \stackrel{\mathrm{def}}{=} M_p$ a fixed $G_0 = D_p$ module. Assume product-one $\Leftrightarrow \boldsymbol{g} \in (G_t)^4 \cap \mathbf{C}_{2^4}$ over ${}_0\boldsymbol{g}$ to be in $\mathrm{Ni}(G_k, \mathbf{C}_{2^4})$ (generation automatic). Then, $G_k = D_{p^{k+1}}$.

Definition: $\boldsymbol{g} \in \mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$ is a (product-one) D_p-iteration of f_0, $k+1$ times. Warning: We aren't composing f_0, $k+1$ times, nor is the iteration unique. Yet, the result is composition of $k+1$ rational functions

determined by f_0 and $\boldsymbol{g}$.

§I.D. Growth of p-cusps with levels from a Spire

Higher Schur-Zassenhaus: A g-p' cusp (here $(\boldsymbol{g}_{\text{H-M}})$**sh**) at level 0 $\Longrightarrow$ a projective sequence of g-p' cusps (here $\{({}_k\boldsymbol{g}_{\text{H-M}})\textbf{sh} \in \text{Ni}_k\}_{k=0}^{\infty}$).

Theorem [Fr07b, Princ. 3.3]: In any Nielsen class, if $\boldsymbol{g}_{\text{H-M}}$ is a p-cusp, so is ${}_k\boldsymbol{g}_{\text{H-M}}$. Inductively in k, $\exists$ a *new* p-cusp at level k (no p-cusp below it): the cusp of $(({}_{k+1}\boldsymbol{g}_{\text{H-M}})q_2^{p^k})$**sh**.

Conclude: # of p-cusps grows with k in a braided *spire* (See §II.H; Picture App. A_2).

§I.E. 3 reasons to pinpoint Hurwitz space components

- Know precise definition field of components (from the B(ranch) C(ycle) L(emma)). Hone in on those over $\mathbb{Q}$.

- Know which components support a Modular Tower: analog for Hurwitz spaces of Shimura variety towers (both generalizing modular curve towers).

- Can identify components by their cusps, enabling Serre's arguments on modular curves for his O(pen) I(mage) T(heorem).

Reminder: When r (number of branch points) is 4, all reduced Hurwitz space components are upper half-plane quotients by a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ [BF02, Prop. 2.3].

§II. Modular curve-like spaces with A_ns replacing D_ps

Liu-Osserman: $\mathrm{Ni}_{(\frac{n+1}{2})^4}$: $G = A_n, n \equiv 5 \bmod 8$

$D_p \leftrightarrow A_n$, $N_p \leftrightarrow S_n$, $\mathrm{C}_2 \leftrightarrow \mathrm{C}_{\frac{n+1}{2}}$: class of $\frac{n+1}{2}$-cycle.

General case of Liu-Osserman is of genus 0 covers and pure(one length > 1 disjoint)-cycle conjugacy classes.

Basic facts: Suppose $\boldsymbol{g} \in A_n^4 \cap \mathbf{C}_{(\frac{n+1}{2})^4}$ satisfying product-one, with $\langle \boldsymbol{g} \rangle$ transitive. Following hold:

- $\langle \boldsymbol{g} \rangle = A_n$; and unless $\boldsymbol{g}$ is **sh** of an $\mathrm{H\text{-}M}$ rep., cusp orbit $\mathrm{Cu}_4(\boldsymbol{g})$ is *pure-cycle* ($g_2 g_3$ itself is pure-cycle).
- g-$2'$ cusps have width 1 or 2 and are represented by **sh** applied to an $\mathrm{H\text{-}M}$ rep. [Fr07b,Prop. 3.10]+[Wm73]: Because $\langle g_2, g_3 \rangle$ and $\langle g_1, g_4 \rangle$ are A_k, $k \geq 4$, for some $k \leq n$, so not $2'$, unless they are cyclic groups.

§II.A. **sh**-incidence for $\mathrm{Ni}^{\mathrm{abs}}_{(\frac{n+1}{2})^4}$

Notation: $x_{i,j} = (i\; i{+}1\; \cdots\; j)$. List of inner $\mathrm{H\text{-}M}$ reps:

$$\mathrm{H\text{-}M}_1 \stackrel{\mathrm{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

$$\mathrm{H\text{-}M}_2 = (\mathrm{H\text{-}M}_1)q_1 \stackrel{\mathrm{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

Absolute (resp. inner) cusps represented by Cu_4 orbits in $\mathrm{Ni}_{(\frac{n+1}{2})^4}/S_n$ (resp. $\mathrm{Ni}_{(\frac{n+1}{2})^4}/A_n$). As with modular curves, $\mathcal{Q}'' = \langle \mathbf{sh}^2, q_1 q_3^{-1} \rangle$ is trivial on Nielsen classes. Cusp orbit of $\mathrm{H\text{-}M}_1$:

$$\{\mathrm{H\text{-}M}_{1,t} = (x_{\frac{n+1}{2},1}, x_{1+t,\frac{n+1}{2}+t}, x_{\frac{n+1}{2}+t,n+t}, x_{n,\frac{n+1}{2}})\}_{t=0}^{n-1}.$$

$(\text{H-M}_{1,t})$**sh**, labeled by middle product $\textbf{mpr}_{\boldsymbol{g}} = \text{ord}(g_2g_3)$:

$$0 \le t \le \tfrac{n-1}{2}, (\text{H-M}_{1,t})\textsf{sh} = [2t{+}1]_1 :$$
$$= (x_{1+t,\frac{n+2t+1}{2}}, (x_{\frac{n+2t+1}{2},n}, x_{1,t}), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$$

$$t' = n{-}1{-}t, \tfrac{n+1}{2} \le t \le n{-}1, (\text{H-M}_{1,n-t'})\textsf{sh} = [2t'{+}1]_2 :$$
$$= ((x_{n-t'+1,n}\, x_{1,\frac{n-2t'+1}{2}}), x_{\frac{n-2t'+1}{2},n-t'}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$$

For $0 \le t \le \frac{n-1}{2}$, $[2t{+}1]_1$ gives a list of absolute cusp reps.

So, we can label absolute cusps as ${}_{\mathsf{c}}O_{2t+1}$, $0 \le t \le \frac{n-1}{2}$.

We list cusps in descending width along the rows and columns. In absolute sh-incidence matrix: $\frac{n+1}{2}$ 1's along 1-1 a(nti)-(sub)d(iagonal); $\frac{n-1}{2}$ 2's along 3-3 ad, etc.

§II.B. **sh**-incidence: $r = 4$ and $\mathrm{Ni}_{34}^{\mathrm{in,abs}}$, $n = 5$

Cusp orbit	$_{\mathbf{c}}O_5$	$_{\mathbf{c}}O_3$	$_{\mathbf{c}}O_1$
$_{\mathbf{c}}O_5$	2	2	1
$_{\mathbf{c}}O_3$	2	1	0
$_{\mathbf{c}}O_1$	1	0	0

Three cusps: Along each row or column the sum is the cusp width —order of ramification of the cusp over $j = \infty$.

- $\bar{\mathcal{H}}^{\rm in,rd}_{(\frac{n+1}{2})^4} \xrightarrow{\text{deg}=2} \bar{\mathcal{H}}^{\rm abs,rd}_{(\frac{n+1}{2})^4} \xrightarrow{\text{deg} = (\frac{n+1}{2})^2} \mathbb{P}^1_j$ ($^-$ includes cusps).
- $\bar{\mathcal{H}}^{\rm abs,rd}_{(\frac{n+1}{2})^4}$ embeds in $\mathbb{P}^1_{j_w} \times \mathbb{P}^1_{j_z}$ (not modular curve, App. C_2).
- $n = 13$: Two (resp. one) width 13, 11, 5, 3 (resp. 18, 14, 2) cusps: No 2-cusps ($_{\mathbf{c}}O_1 = (\boldsymbol{g}_{\text{H-M}})$**sh**; rest o(nly)-$2'$ cusps).

§II.C. Abs-inn sh-incidence for $n = 13$, $\text{Ni}^{\rm in,rd}_{74}$

Cusp orbit	$_{\mathbf{c}}O_{13}$	$_{\mathbf{c}}O_{11}$	$_{\mathbf{c}}O_{9}$	$_{\mathbf{c}}O_{7}$	$_{\mathbf{c}}O_{5}$	$_{\mathbf{c}}O_{3}$	$_{\mathbf{c}}O_{1}$
$_{\mathbf{c}}O_{13}$	$\frac{0\|2}{2\|0}$	$\frac{1\|1}{1\|1}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{0\|2}{2\|0}$	$\frac{1\|1}{1\|1}$	$\frac{1}{1}$
$_{\mathbf{c}}O_{11}$	$\frac{1\|1}{1\|1}$	$\frac{0\|2}{2\|0}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{1\|1}{1\|1}$	$\frac{0\|1}{1\|0}$	$\frac{0}{0}$
$_{\mathbf{c}}O_{9}$	$2\|2$	$2\|2$	4^0	4	$1\|1$	$0\|0$	0
$_{\mathbf{c}}O_{7}$	$2\|2$	$2\|2$	4	2^1	$0\|0$	$0\|0$	0
$_{\mathbf{c}}O_{5}$	$\frac{0\|2}{2\|0}$	$\frac{1\|1}{1\|1}$	$\frac{1}{1}$	$\frac{0}{0}$	$\frac{0\|0}{0\|0}$	$\frac{0\|0}{0\|0}$	$\frac{0}{0}$
$_{\mathbf{c}}O_{3}$	$\frac{1\|1}{1\|1}$	$\frac{0\|1}{1\|0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0\|0}{0\|0}$	$\frac{0\|0}{0\|0}$	$\frac{0}{0}$
$_{\mathbf{c}}O_{1}$	$1\|1$	$0\|0$	0	0	$0\|0$	$0\|0$	0

§II.D. Genus $\mathbf{g}_{n,*}, * = abs/in$: [Fr07b,Prop. 5.15]

- Each col. sums to cusp ram. index over $\mathbb{P}^1_j$: cusp col.s over abs. cusp ${}_{\mathbf{c}}O_{13}$ sum to 13; only cusp col. over abs. cusp ${}_{\mathbf{c}}O_9$ sums to 18.

- Only $\gamma_0 = q_1q_2$ (resp. $\gamma_1 = \mathbf{sh}$) fixed points in ${}_{\mathbf{c}}O_9$ (resp. ${}_{\mathbf{c}}O_7$) cols. indicated by superscript 0 (resp. 1). sh-incidence matrix same as γ_0-incidence matrix - elliptic fixed points forced.

- $\mathbf{g}_{5,\text{abs}} = \mathbf{g}_{5,\text{in}} = 0$; $\mathbf{g}_{13,\text{abs}} = 1, \mathbf{g}_{13,\text{in}} = 3$; Prop. 5.15 has all $n \equiv 1 \mod 4$. For $n \equiv 1 \mod 8$, two components conjugate over $L/\mathbb{Q}$ quadratic.

II.E. Modular curve analog $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ and (product-one) A_n-iterations

Changing D_p to A_n, for prime $p|n!$, but $p \not| \frac{n+1}{2}$. Want G_0 module $M_{n,p}$ and p-Frattini covers $G_{k,\mathrm{ab}} \to A_n = G_0$ so $\ker(G_{k,\mathrm{ab}} \to G_{k-1,\mathrm{ab}}) = M_{n,p}$.

$\exists$ universal (abelianized) p-Frattini cover $\tilde\varphi : \tilde G_{ab}(A_n) \to A_n$.

$\ker(\tilde\varphi)$: fin. dim. tors.-free $\mathbb{Z}_p[A_n]$ module, $m_{n,p} = \ker(\tilde\varphi)/p\ker(\tilde\varphi)$.

Defining levels: $G_{k,\mathrm{ab}} = \tilde G_{ab}(A_n)/p^k \ker(\tilde\varphi)$: Component $\mathcal{H}'_k$ from a braid orbit Ni'_k on

$$\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}} \leftrightarrow Y_1(p^{k+1}).$$

As (D_p, p) is to $\mathbb{Z}/p$, (A_n, p) is to $M_{n,p}$.

$(A_n, \mathbf{C}_{(\frac{n+1}{2})^4}, p = 2)$ Modular Tower: Projective sequence $\{\mathcal{H}'_k\}_{k=0}^{\infty}$.

§II.F. The Spin_n-lifting invariant

When does there exist at least one MT over $\mathcal{H}(A_n, \mathbf{C}, p)$ for a general $\mathbf{C}$?

When is there more than one?

When are all levels of a MT defined over $\mathbb{Q}$?

Odd $d_1 \le d_2 \le \cdots \le d_r$ pure-cycle A_n lengths, $I_{\boldsymbol{d}} \stackrel{\text{def}}{=} \sum_{i=1}^r d_i - 1$. [Fr+Se+We] Liu-Osserman case $I_{\boldsymbol{d}} = 2(n-1)$ ($\mathcal{H}(A_n, \mathbf{C}_{d_1 \cdots d_r})$ irreducible):

For $p \ne 2$, at least one MT. For $p = 2$, at least one iff

(*) $\sum_{i=1}^r \frac{d_i^2 - 1}{8} \equiv 0 \mod 2$ (includes if there is a H-M rep.).

§II.G. Listing 2-cusps in $\mathrm{Ni}(G_1, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$.

[Talk$_3$] explains: If $I_{\boldsymbol{d}} > 2(n-1)$ usually more than one component at level 0 and then at least one supports a (nonempty) MT for all allowed primes.

Call level 0 cusp ${}_{\mathbf{c}}O_{\boldsymbol{g}}$, $\boldsymbol{g} \in \mathrm{Ni}_{(\frac{n+1}{2})^4}$, a 2-T(otal)J(ump) cusp if all cusp orbits on $\mathrm{Ni}(G_1, \mathbf{C}_{(\frac{n+1}{2})^4})$ over it are 2-cusps.

Thm: ${}_{\mathbf{c}}O_{\boldsymbol{g}}$ is 2-TJ iff the middle product $\mathbf{mpr}_{\boldsymbol{g}}$ (p. 9) satisfies

$$(-1)^{\frac{\mathbf{mpr}_{\boldsymbol{g}}^2 - 1}{8}} = -1.$$

Conclusion: In p. 10 listing of cusp reps. by middle products: $0 \le t \le \frac{n-1}{2}$, $[2t+1]_1$, 2-cusps correspond to

$$(-1)^{\frac{(2t+1)^2-1}{8}} = -1, \text{ or } t \equiv 3, 5 \mod 8.$$

§II.H. Why an H-M Modular Tower, $\{\mathcal{H}'_k\}_{k=0}^{\infty}$ – each level has an H-M rep. – has a spire

Since $[n]_1$ is an H-M rep., an adjustment to Thm. of §I.D gives a spire on this MT (App. A_2). Conclude: 2-cusps grow with k on each MT over $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, $n \equiv 5 \mod 8$.

Idea: Combine these two facts:

- For $\boldsymbol{g} \in \mathrm{Ni}(G_{i'}, \mathbf{C})^{\mathrm{in,rd}}$ representing a p cusp, with $p^{j'} || \mathbf{mpr}_{\boldsymbol{g}}$, then for any $\boldsymbol{g}' \in \mathrm{Ni}(G_{i'+k'}, \mathbf{C})^{\mathrm{in,rd}}$ over $\boldsymbol{g}$, $p^{j'+k'} || \mathbf{mpr}_{\boldsymbol{g}}$.
- If $\boldsymbol{g}'$ is an H-M rep. (so $\boldsymbol{g}$ is also), then there is an explicit braid $(\boldsymbol{g}')q = \boldsymbol{g}''$, with $p || \mathbf{mpr}_{\boldsymbol{g}''}$.

App. §A_2: A modular curve-like Spire

Theorem 1. *A* MT *with a H(arbater)-M(umford) cusp branch of p-cusps has a spire: sub-tree isomorphic to a modular curve cusp tree. Holds for $p = 2$ at level 1, for* Liu-Osserman $n \equiv 5 \mod 8$. *Doesn't hold for* $n \equiv 1 \mod 8$.

SPIRE: Growth of p cusps with level: Subscript is power of p dividing the middle product.

$$\begin{array}{rllll}
\text{Level 1}: & \bullet_p & & & \\
\text{Level 2}: & \bullet_{p^2} & \bullet_p & & \\
\text{Level 3}: & \bullet_{p^3} & \bullet_{p^2} & \bullet_p & \\
\ldots: & \ldots & \ldots & \ldots & \ldots
\end{array}$$

App. B_2. p-cusp MT Conjectures

Main Conj. 1: K a number field, then High MT levels have no K points. Cadoret [Ca08]: Showed Strong Torsion Conjecture on abelian varieties implies this.
Cadoret-Tamagawa [CaT08] showed this for $r \leq 4$.

Main Conj. 2: a. High MT levels have general type (for $r = 4$ implies Conj. 1) and are b. relatively p-Frattini over the previous level ($\Longrightarrow$ weak form of Serre's O(pen) I(mage) T(heorem)). This talk shows the a. part for MTs over Liu-Osserman spaces, but hasn't yet shown the b. part.

15 years ago Pierre Debes and I noted relation of dihedral groups to work of Kamienny and Mazur [DFr94].

Compare to Mazur-Meryl

Conjecture: There can't be regular involution realizations over $\mathbb{Q}$ of all $D_{p^{k+1}}$, k unbounded, p any (fixed) odd prime, with at most r_0 (any number you like), branch points $\Leftrightarrow$ a uniform bound on $\mu(p^{k+1})$-torsion points on hyperelliptic jacobians of dimension no more than r_0: with such (*involution*) realizations necessarily in $\mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^{r_0}})^{\mathrm{in,rd}}$.

The MT program generalizes that: Replacing dihedral groups by any finite G, p any prime with G p-perfect (has no $\mathbb{Z}/p$ quotient). To date, only a handful of involution realizations of dihedral groups known —most from Mazur's theorem, even without a bound (r_0) on branch points.

App. C_2: Andre's Thm. and Shimura special points

Each space $\bar{\mathcal{H}}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4}$, $n \equiv 1 \mod 4$ embeds in $\mathbb{P}^1_{j_w} \times \mathbb{P}^1_{j_z}$: Each cover with branch points $\{z_1, \dots, z_4\}$, has a distinguished point w_i over z_i corresponding to the pure-cycle, $i = 1, \dots, 4$, akin to modular curve case.

The geometric monodromy group of $\bar{\mathcal{H}}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4} \to \mathbb{P}^1_{j_z}$ is $A_{(\frac{n+1}{2})^2}$ (easily seen from the pure-cycle cusps described § II.A). So, $\bar{\mathcal{H}}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4}$ cannot possibly be a modular curve.

The points of form $(j(\tau'), j(\tau''))$ with τ' and τ'' fixed by "complex multiplications" are *(Shimura) special* [Sh71, §4.4]. Andre's Thm: There are only finitely many such points on a non-modular curve embedded in $\mathbb{P}^1_{j_w} \times \mathbb{P}^1_{j_z}$.

Bibliography:

- [BiF86] R. Biggers and M. D. Fried, *Irreducibility of moduli spaces of cyclic unramified covers of genus g curves*, TAMS **295** (1986), 59–70.
- [BF02] P. Bailey and M. D. Fried, *Hurwitz monodromy, spin separation and higher levels of a modular tower*, Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), Proc. Sympos. Pure Math., vol. 70, Amer. Math. Soc., Providence, RI, 2002, pp. 79–220. MRMR1935406 (2005b:14044)
- [Ca08] A. Cadoret, *Rational points on Hurwitz towers*, preprint as of Jan. 2008, 1–30.
- [CaT08] A. Cadoret and A. Tamagawa, Uniform boundedness of p-Torsion of Abelian Schemes, preprint as of June 2008, 1–25.
- [D06] P. Dèbes, *Modular Towers: Construction and Diophantine Questions*, same vol. as [**?**].
- [DE06] P. Dèbes and M. Emsalem, *Harbater-Mumford Components and Towers of Moduli Spaces*, Journal de l'Institut Mathématique de Jussieu, **5/03** (2006), 351–371.
- [DFr94] P. Débes and M. Fried, *Nonrigid situations in constructive Galois theory*, Pacific Journal **163** (1994), 81–122.

Fr95 M. Fried, *Introduction to Modular Towers: Generalizing dihedral groupmodular curve connections*, Recent Developments in the Inverse Galois Problem, Cont. Math., proceedings of AMS-NSF Summer Conference 1994, Seattle **186** (1995), 111-171.

- [Fr06] M. D. Fried, *The Main Conjecture of Modular Towers and its higher rank generalization*, in *Groupes de Galois arithmetiques et differentiels* (Luminy 2004; eds. D. Bertrand and P. Dèbes), Seminaires et Congres, Vol. **13** (2006), 165–233.
- [Fr07a] M. Fried, *Alternating groups and moduli space lifting invariants*, Out for refereeing (2007), 1–36, at http://www.math.uci.edu/˜mfried/paplist-mt/hf-can0611591.pdf.

- [Fr07b] M. Fried, *Connectedness of families of sphere covers of A_n-Type*, 1–56, preprint 04/29/08 http://www.math.uci.edu/~mfried/paplist-mt/twoorbit.pdf.
- [FK97] M. Fried and Y. Kopeliovic, *Applying Modular Towers to the inverse Galois problem*, Geometric Galois Actions II Dessins d'Enfants, Mapping Class Groups and Moduli, vol. 243, Cambridge U. Press, 1997, London Math. Soc. Lecture Notes, pp. 172–197.
- [FV91] M. D. Fried and H. Völklein, *The inverse Galois problem and rational points on moduli spaces*, Math. Ann. **290** (1991), no. 4, 771–800. MRMR1119950 (93a:12004)
- [FV92] M. D. Fried and H. Völklein, *The embedding problem over an Hilbertian-PAC field*, Annals of Math **135** (1992), 469-481.
- [Mu72] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, Comp. Math. **24** (1972), 129–174.
- [Ser90] J.-P. Serre, *Relèvements dans $\tilde{A}_n$*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 8, 477–482. MRMR1076476 (91m:20010)
- [Ser92] J.-P. Serre, *Topics in Galois Theory*, Res. Notes in Math. Notes written by H. Darmon, Jones and Bartlett, 1992.
- [Ser96] J.-P. Serre, *Galois Cohomology*, 4th Edition, Springer: Translated from the French by Patrick Ion; original from Springer LN5 (1964), 1996.
- [Se68] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, 1st ed., McGill University Lecture Notes, Benjamin, New York • Amsterdam, 1968, in collaboration with Willem Kuyk and John Labute.
- [Sh71] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Pub. of Math. Soc. of Japan **11**, Princeton U. Press, 1971.

- [Talk$_1$] Extending www.math.uci.edu/~mfried/talklist-mt/London1-ModCurves.pdf.
- [Talk$_3$] www.math.uci.edu/~mfried/talklist-mt/London2-AltGps.pdf.
- [Vo96] H. Völklein, *Groups as Galois Groups*, an Introduction, Cambridge Studies in Mathematics 1996 **53**, ISBN 0-521-56280-5.

- [WS] Modular Tower web site: **http://www.math.uci.edu/˜mfried** → §I.a, 5th line down [Articles: Generalizing modular curve properties to Modular Towers] and → §I.b, 3rd listing down: Definitions: Modular Towers and Profinite Geometry.
- [Wei05] T. Weigel, *Maximal ℓ-frattini quotients of ℓ-poincare duality groups of dimension 2*, volume for O. H. Kegel on his 70th birthday, Arkiv der Mathematik–Basel, 2005.
- [Wm73] A. Williamson, *On primitive permutation groups containing a cycle*, Math. Zeit. **130** (1973), 159–162.