How pure-cycle Nielsen classes Test the Main Modular Tower Conjecture

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Tight connections between three arithmetic problems MT/RIGP/STC:

- M(ain)C(onjecture) on Modular Towers (MTs),
- R(egular)I(nverse)G(alois)P(roblem), and the
- S(trong)T(orsion)C(onjecture) on abelian varieties.

MT Main Conjecture explicitly challenges the STC

MTs encodes a huge portion of the RIGP into questions about towers of varieties (MTs). A simple *ramification assumption* on regular realizations forces K points at all tower levels on some MT. Generalization of Mazur-Merel implies this should be impossible: STC $\implies$ MC (Cadoret, [STMT]). So, if STC holds, the ramification assumption must be wrong.

I'll show the MC holds for ∞-ly many (non-modular curve) MTs using the Fried-Serre lifting invariant. Technique: Explicitly analysis projective systems of cusps on a MT cusp tree. We will see geometrically why it holds in these cases, giving info about what is needed to prove the general case. So, these cases challenge the STC, about which little is known.

Part I: Conjugacy classes and covers

G a group, **C** is r conjugacy classes in G.

- $\boldsymbol{g} = (g_1, \dots, g_r) \in \mathbf{C}$ means $g_{(i)\pi}$ is in C_i, for some π permuting $\{1, \dots, r\}$.
- $\Pi(\boldsymbol{g}) \stackrel{\text{def}}{=} \prod_{i=1}^{r} g_i$ (order matters).

An analytic cover, $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces, ramifies over a finite set of points $\boldsymbol{z} = z_1, \dots, z_r \subset \mathbb{P}^1_z$: $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}$.
Then, $\varphi \implies (G, \mathbf{C}, \boldsymbol{z}), G \le S_n$, with $n = \deg(\varphi)$: G the *monodromy group of* φ.

Nielsen classes/ R(iemann's)E(xistence)T(heorem)

Fix $\boldsymbol{z} = \boldsymbol{z}^0$ and *classical generators* of $\pi_1(U_{\boldsymbol{z}^0}, z_0)$.

Combinatorial description of all $\varphi \implies (G, \mathbf{C})$:

Nielsen classes:

$$\{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \Pi(\boldsymbol{g}) = 1\} \stackrel{\text{def}}{=} \mathrm{Ni}(G, \mathbf{C}).$$

Projective r space $\mathbb{P}^r \Leftrightarrow$ degree $\leq r$, monic polynomials; deg $< r - 1$ or with equal zeros form its *discriminant* locus D_r. Denote $\mathbb{P}^r \setminus D_r$ by U_r.

Hurwitz combinatorics: Deformations (r branch points) of $\varphi \implies$ paths in U_r based at $\boldsymbol{z}^0$.

One cover defines a family: $\varphi : X \to \mathbb{P}^1_z \implies$

1. Permutation representation of $\pi_1(U_r, \boldsymbol{z}^0) \stackrel{\text{def}}{=} H_r$ *Hurwitz monodromy* on orbit Ni'_φ — independent of classical generators — of $[\varphi] \in \mathrm{Ni}(G, \mathbf{C})$.

2. An unramified connected cover $\mathcal{H}(G, \mathbf{C})_\varphi \to U_r$: Hurwitz space component containing φ.

Equivalences of covers and Nielsen classes.

[Abs.]$\varphi' : X' \to \mathbb{P}^1_z \sim \varphi \Leftrightarrow \boldsymbol{g} = h\boldsymbol{g}'h^{-1}, h \in N_{S_n,\mathbf{C}}(G)$.

[Inn.]φ Galois with $\mu : \mathrm{Aut}(X/\mathbb{P}^1_z) \stackrel{\text{isom}}{\longrightarrow} G \sim (\varphi', u') \Leftrightarrow$ $\boldsymbol{g} = h\boldsymbol{g}'h^{-1}, h \in G$.

Part II: Importance of Connectedness Results:

II.A. Constellations of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{abs}}$ [AGLI, §1]

$\xrightarrow{g\geq 1}$	$\ominus \oplus$	$\ominus \oplus$	. . .	$\ominus \oplus$	$\ominus \oplus$	$\xleftarrow{1\leq g}$
$\xrightarrow{g=0}$	$\ominus$	$\oplus$	. . .	$\ominus$	$\oplus$	$\xleftarrow{0=g}$
$n \geq 4$	$n = 4$	$n = 5$	. . .	n even	n odd	$4 \leq n$

Theorem 1 (tag $\xrightarrow{g=0}$, $r = n-1$, $n \geq 5$). $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\text{in}}$ *has one component. Further,* $\Psi^{\text{in}}_{\text{abs}} : \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\text{in}} \to \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\text{abs}}$ is deg. 2.

Theorem 2 (tag $\xrightarrow{g\geq 1}$, $r \geq n \geq 5$). $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\text{in}}$ *has two components,* $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\text{in}}$ *(symbol* $\oplus$*) and* $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\text{in}}$ *(symbol* $\ominus$*). Further*

$$\Psi^{\text{in},\pm}_{\text{abs}} : \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\text{in}} \to \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\text{abs}} \text{ has degree 2.}$$

For $n = 4$, two 3-cycle classes C_{+3}, C_{-3} in A_4, $\mathbf{C} = \mathbf{C}_{+3^{s_1} \cdot -3^{s_2}}$: $\mathrm{Ni}(G, \mathrm{C}_{\pm 3^{s_1,s_2}})$ nonempty iff

$$s_1 - s_2 \equiv 0 \mod 3 \text{ and } s_1 + s_2 = r.$$

Frattini covers

Frattini cover $G' \to G$ is a group cover (surjection) with restriction to a proper subgroup not a cover. Get a lifting invariant from a *central* Frattini cover.

Central Frattini from A_n: Spin_n^+ the nonsplit degree 2 cover of the connected component O_n^+ of the orthogonal group. Regard $S_n \subset O_n$; $A_n \subset O_n^+$. Denote pullback of A_n to Spin_n^+ by Spin_n. Identify $\ker(\mathrm{Spin}_n \to A_n)$ with $\{\pm 1\}$.

F-S Small lifting invariants ([LUM,§1], [Ser90a])

Odd order $g \in A_n$ has a unique odd order lift, $\hat{g} \in \mathrm{Spin}_n$. Let $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C})$ with $\mathbf{C}$ odd-order. *Small* lifting invariant:

$$s(\boldsymbol{g}) = s_{\mathrm{Spin}_n}(\boldsymbol{g}) = \hat{g}_1 \cdots \hat{g}_r \in \{\pm 1\}.$$

For g odd-order, let $w(g)$ by the number of cycles in g with lengths (ℓ) with $\frac{\ell^2-1}{8} \equiv 1 \mod 2$.

Theorem 3 (F-S). *On any braid orbit, $s(\boldsymbol{g})$ is constant (explains Const. diag. comps). If genus 0 Nielsen class, then $s(\boldsymbol{g}) = (-1)^{\sum_{i=1}^{r} w(g_i)}$.*

II.B. Pure-cycle components

- $g \in S_n$ is *pure-cycle* if one cycle has length > 1.
- Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ is *pure-cycle* if all conjugacy classes are pure-cycle (a d-cycle).
- If $d_1, \dots, d_r$ are the pure-cycle lengths, denote the Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_r})^*$ (* an equivalence).

Assume $G \le S_n$ transitive and $\mathbf{C}^{S_n} \stackrel{\mathrm{def}}{=} \mathbf{C}_{d_1 \cdots d_r}$ image of $\mathbf{C}$ in S_n, with d_i s all odd. Necessary condition $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ is nonempty: Genus

$$\mathbf{g} = \mathbf{g}_{d_1 \cdots d_r} \stackrel{\mathrm{def}}{=} \frac{\sum_{i=1}^{r} d-1}{2} - (n-1) \text{ is non-negative.}$$

Liu-Osserman genus 0 result [LOs06]

Theorem 4. *If* $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_r})$ *has genus 0, then* $G = A_n$*, and* H_r *is transitive on it.*

Compactify the reduced inner space:

$$\bar{\mathcal{H}}(A_n, \mathbf{C}_{d_1 \cdot d_2 \cdot d_3 \cdot d_4})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \bar{\mathcal{H}}_{n, d_1 \cdots d_4}.$$

Consider $\{\bar{\mathcal{H}}(G_k(A_n), \mathbf{C}_{d_1 \cdot d_2 \cdot d_3 \cdot d_4})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \bar{\mathcal{H}}_{n, d_1 \cdots d_4, k}\}_{k=0}^{\infty}$ with $G_k(A_n) \to A_n$ the universal exponent 2^k 2-group extension of A_n.

Statement of the Goal

Goal ($r = 4$): Given a projective sequence of components $\{\bar{\mathcal{H}}'_{n,d_1\cdots d_4,k}\}_{k=0}^{\infty}$ on $\{\bar{\mathcal{H}}_{n,d_1\cdots d_4,k}\}_{k=0}^{\infty}$ (defined uniformly over some number field), decide if genus of level k grows with k.

Up to Appendix, assume all d_is the same $(= d)$. Genus 0 Nielsen class implies $\implies 2(d-1) = n-1$.

Inner (resp. absolute) Reduced spaces [BFr02, §2]

Reduced equiv.: $\varphi : X \to \mathbb{P}^1_z \sim \beta \circ \varphi, \beta \in \mathrm{PGL}_2(\mathbb{C})$.

j-invariant: $\boldsymbol{z} \in U_4 \mapsto j_{\boldsymbol{z}} \in U_\infty \stackrel{\text{def}}{=} \mathbb{P}^1_j \setminus \{\infty\}$ of $\boldsymbol{z}$.

Normalize so $j = 0$ and 1 are *elliptic points*: $j_{\boldsymbol{z}}$ with more than a Klein 4-group stabilizer in $\mathrm{PGL}_2(\mathbb{C})$.

Reduced classes of covers with j-invariant $j' \in U_\infty$ $\Leftrightarrow$ elements of reduced Nielsen classes.

Part III: $r = 4$ Upper-half plane quotients

Recall: $H_4 = \langle q_1, q_2, q_3 \rangle$: Acts on any Nielsen classes with $r = 4$ by a twisting on its 4-tuples:

$$q_2 : \boldsymbol{g} \mapsto (\boldsymbol{g})q_2 = (g_1, g_2 g_3 g_2^{-1}, g_2, g_4).$$

Reduced equivalence corresponds to modding out the Nielsen class by $\mathcal{Q}'' = \langle (q_1 q_2 q_3)^2, q_1 q_3^{-1} \rangle \le H_4$.

H_4 on reduced Nielsen classes factors through the *mapping class group*: $\bar{M}_4 \stackrel{\text{def}}{=} H_4/\mathcal{Q}'' \equiv \mathrm{PSL}_2(\mathbb{Z})$.

III.A. Using generators of $\bar{M}_4$

$\bar{M}_4 = \langle \gamma_0, \gamma_1, \gamma_\infty \rangle, \gamma_0 = q_1 q_2$ (order 3),
$\gamma_1 =$ **shift** $= q_1 q_2 q_3$ (order 2),
$\gamma_\infty = q_2$ ($j = \infty$ monodromy generator),
satisfying the product-one relation: $\gamma_0 \gamma_1 \gamma_\infty = 1$.
The *cusp group* $\text{Cu}_4 = \langle q_2, \mathcal{Q}'' \rangle \le H_4$:
A cusp is an orbit of Cu_4. $(\boldsymbol{g})$**sh** $\mapsto$ reduced class of (g_2, g_3, g_4, g_1). and **sh**^{2} is trivial.

Riemann-Hurwitz on components

Interpret R-H: Denote $(\gamma_0, \gamma_1, \gamma_\infty)$ acting on Ni_{d^4} as giving branch cycles for $\bar{\mathcal{H}}_{d^4} \to \mathbb{P}^1_j$. Denote the resulting permutations by $(\gamma'_0, \gamma'_1, \gamma'_\infty)$:

- Points over 0 (resp. 1) $\Leftrightarrow$ orbits of γ_0 (resp. γ_1).
- The index contribution $\mathrm{ind}(\gamma_\infty)$ from a cusp with rep. $\boldsymbol{g} \in \mathrm{Ni}_{d^4}$ is $|(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}''| - 1$.

2-Frattini extensions of A_5

$(\mathbb{Z}/2)^2 \times^s \mathbb{Z}/3 = A_4$: The universal 2-Frattini extension of A_4 is ${}_2\tilde{G}(A_4) = \tilde{F}_2 \times^s \mathbb{Z}/3$.

Univ. 2-Frattini extension ${}_2\tilde{G}(A_5)$ of A_5: Restriction over A_4 is ${}_2\tilde{G}(A_4)$. With
$\ker_0 = \ker({}_2\tilde{G}(A_5) \to A_5)$,
$\Phi_1(\ker_0) = \langle(\ker_0, \ker_0), \ker_0^2\rangle$.
Then, $\Phi_k(\ker_0) \stackrel{\text{def}}{=} \Phi_{k-1}(\Phi_1(\ker_0))$.

Iterate Φ_1 to get max. exp. 2^k Frattini extension of A_5: $G_k(A_5) \stackrel{\text{def}}{=} {}_2\tilde{G}(A_5)/\Phi_k(\ker_0)$.

III.B. Modular curve-like towers

$$\{\bar{\mathcal{H}}(G_k(A_5), \mathbf{C}_{3^4})^{\text{in,rd}}\}_{k=0}^{\infty}$$

Ram_{r_0}: Choose any r_0. For $k \geq 0$, use covers in $\text{Ni}(G_k, \mathbf{C}_k)$ with at most r_0 classes in $\mathbf{C}_k$.

Question 5 (RIGP(A_5,p=2,r_0) Quest.). Is there r_0, so the RIGP holds for all G_k s from covers in Ram_{r_0}?

Theorem 6. *If the answer is "Yes!," then there are $2'$ conjugacy classes* $\mathbf{C}$ *(no more than r_0) in G, and a projective system $\{\mathcal{H}'_k \subset \mathcal{H}(G_k, \mathbf{C})^{\text{in,rd}}\}_{k=0}^{\infty}$ (a Modular Tower component branch over $\mathbb{Q}$) each having a $\mathbb{Q}$ point ([D06] [FrK97]).*

The Main Conjecture

Conjecture 7 (MainConj.). If $k >> 0$, $\mathcal{H}'^{\rm rd}_k(\mathbb{Q}) = \emptyset$.

Our examples: Towers over $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$, odd $n \geq 5$, $p = 2$. Three cusp types [LUM, §3]:

$$H_{2,3}(\boldsymbol{g}) \stackrel{\rm def}{=} \langle g_2, g_3 \rangle \text{ and } H_{1,4}(\boldsymbol{g}) = \langle g_1, g_4 \rangle;$$

and $(\boldsymbol{g})\mathbf{mpr} \stackrel{\rm def}{=} \mathrm{ord}(g_2 g_3)$, *middle product order*.

- *p cusps*: $p \big| (\boldsymbol{g})\mathbf{mpr}$.
- *g(roup)-p'*: $H_{2,3}(\boldsymbol{g})$ and $H_{1,4}(\boldsymbol{g})$ are p' groups.
  H-M rep.: $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1}) \implies (\boldsymbol{g})\mathbf{sh}$ is g-p'.
- *o(nly)-p'*: $p \not| (\boldsymbol{g})\mathbf{mpr}$, but the cusp is not g-$p'$.

III.C. **sh**-incidence for $r=4$ and $\mathrm{Ni}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4}$

$(\boldsymbol{g})$**mpr**: (g_2, g_3) pairs for abs. cusp reps.:

n: H-M rep.: $(\bullet, (1 \dots \frac{n+1}{2}), (\frac{n+1}{2}\, \frac{n+3}{2} \dots n), \bullet)$

$n{-}2$: $(\bullet, (2 \dots \frac{n-1}{2}\, \frac{n+3}{2}\, \frac{n+1}{2}), (\frac{n+1}{2}\, \frac{n+3}{2} \dots n), \bullet)$

$\dots$

1: shift of H-M rep.: $(\bullet, (\frac{n+1}{2}\, \frac{n+3}{2} \dots n)^{-1}, (\frac{n+1}{2}\, \frac{n+3}{2} \dots n), \bullet)$

1. Fill in $\bullet$s (1st and last rows hint how), and apply Cu_4.
2. q_2 orbit length is $2 \cdot (\boldsymbol{g})$**mpr** unless $(\boldsymbol{g})$**mpr** $= o$ odd, and $\mathrm{ord}((g_2g_3)^{\frac{o-1}{2}}g_2) = 2$ [BFr02, Prop. 2.17]. Latter for L-O cusps, each is H-M or o-$2'$; widths (top-bottom) $n, n-2, \dots, 1$.

$$\deg(\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}/\mathbb{P}^1_j) = \left(\frac{n+1}{2}\right)^2.$$

See from **sh**-incidence one connected component of genus 0.

sh-incidence Matrix: $r = 4$ and $\mathrm{Ni}^{\mathrm{in,rd}}_{\frac{(n-1)}{2}^4}$

Pairing on Cu_4 orbits: $(O, O') \mapsto |O \cap (O')\mathbf{sh}|$. $O_{5,5;2}$ (resp. $O_{1,2}$) indicates 2nd **mpr** 5, width 5 (resp. only **mpr** 1, width 2) orbit. **sh**-incidence gives $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ genus.

Orbit	$O_{5,5;1}$	$O_{5,5;2}$	$O_{3,3;1}$	$O_{3,3;2}$	$O_{1,2}$
$O_{5,5;1}$	0	2	1	1	1
$O_{5,5;2}$	2	0	1	1	1
$O_{3,3;1}$	1	1	0	1	0
$O_{3,3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

Complete orbit for $\bar{M}_4 = \langle \mathbf{sh}, \gamma_\infty \rangle$ on $\mathrm{Ni}^{\mathrm{in,rd}}_{3^4}$ in 2-steps: Apply $(\mathbf{sh} \circ \mathrm{Cu}_4)^2$ to H-M rep.

Frattini Principles [LUM, §3]

A MT is defined by a projective sequence $\{\mathrm{Ni}'_k\}_{k=0}^{\infty}$ of H_r orbits on $\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in,rd}} \implies$ there is a projective sequence of cusp reps (cusp branch).

[FP1] A p cusp at level k_0 has above it at level k only p cusps of width increased by p^{k-k_0}.

[FP2] g-$2'$ cusp at level 0 $\implies$ g-$2'$ cusp branch.

[FP3] Lifting invariant gives iff test for all cusps above level k o-p' cusps being p cusps ([LUM, §4], [We]).

Cusp Tree Conclusions in Liu-Osserman cases

[STMT] Strong Tors. Conj. $\Longrightarrow$ Main MT Conj. and $(\sim\Leftrightarrow)$.

Apply F-S lift inv. to $(g_2, g_3, (g_2g_3)^{-1})$ for Ni_{34}: Level 0 o-$2'$ cusps $O_{5,5,\bullet}$ and $O_{3,3,\bullet}$ have only 2 cusps above them: $(A_5, \mathbf{C}_{3^4}, p = 2)$ cusp tree has only g-$2'$ or 2 cusp branches.

Theorem 8. *If ≥ 3 p cusps for any MT level k $\Longrightarrow$ Main Conj $\Longrightarrow$ holds for L-O cases (many 2 cusps at level 1). If a cusp branch is both H-M and p, then MT cusp tree contains a spire: a modular curve cusp tree. At level 1, holds for (L-O) $n = 5$, but not for $n = 9$.*

Question 9. When does it hold for Fried + L-O cases?

Appendix A: Using Lifting Invariant on p. 19

List of 3-tuples $(g_2, g_3, (g_2g_3)^{-1})$, with parameter $1 \le k \le \frac{n-1}{2}$:

- $\mathrm{ord}(g_2g_3) = 2k+1$; and $\langle g_2, g_3 \rangle$ is isomorphic to $A_{k+\frac{n+1}{2}}$.

[LUM, Fratt. Princ. 3]: Since 2 part of the Schur multiplier of A_n is just $\mathbb{Z}/2$, all cusps at level 1 above an o-$2'$ cusp are 2-cusps if and only if $s_{\mathrm{Spin}_n/A_n}(g_2, g_3, (g_2g_3)^{-1}) = -1$. Apply F-S formula (p. 9): In each case $(g_2, g_3, (g_2g_3)^{-1})$ has genus 0. So lifting invariant satisfies: $k \implies (-1)^{\frac{(2k+1)^2-1}{2}}$. Example: $n = 9$, $k = 1 \implies -1$, $2 \implies -1$, $3 \implies +1$, $4 \implies +1$.

Appendix B: Why I took all the d_i s equal

Basic Conjecture: A MT whose levels are uniformly defined over one number field is defined by a g-p' cusp branch [LUM, Conj. 1.5] (evidence in [LUM, §4.4].

Group theory: Odd pure-cycles generate an alternating (or cyclic) group $\implies$ a g-$2'$ cusp must be an H-M rep. $\implies$ d_i s equal in pairs. So, dealing with $\{\mathcal{H}_{n,d_1^2\cdot d_2^2,k}\}_{k=0}^{\infty}$.

Case of $\{\bar{\mathcal{H}}'_{n,d_1^2\cdot d_2^2,k}\}_{k=0}^{\infty}$ where $d_1 \neq d_2$. Fact: Genus of $\bar{\mathcal{H}}_{n,d_1^2\cdot d_2^2,0}$ exceeds 0. One possibility: All $\mathcal{H}_{n,d_1^2\cdot d_2^2,k}$ s are the same space. Producing a single 2-cusp, however, at level 1 excludes this: so, the same argument works.

Abbreviated References: [LUM] has much more

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[Def-Lst]Select from the list in www.math.uci.edu/conffiles_rims/deflist-mt/full-deflist-mt.html of present MT-related definitions. 09/05/06 examples: Branch-Cycle-Lem CFPV-Thm Cusp-Comp-Tree FS-Lift-Inv Hurwitz-Spaces Main-MT-Conj Modular-Towers Nielsen-Classes RIGP Strong-Tors-Conj mt-rigp-stc p-Poincare-Dual sh-Inc-Mat. A similar URL, www.math.uci.edu/conffiles_rims/deflist-mt/full-paplist-mt.html, is a repository for not just mine, but also of the growing list of those joining the MT project.

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