Atomic Orbital-type cusps on Alternating Group Modular Towers

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Let $\varphi : X \to \mathbb{P}^1_z$ be a map on compact Riemann surfaces X; φ Galois having group G. Then, φ defines these quantities:

- Unordered branch points $\boldsymbol{z} = \{z_1, \dots, z_r\} \in U_r$: Space of unordered distinct points on $\mathbb{P}^1_z$;
- Conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$ in G; and
- A *Poincaré extension* of groups (abelianized theory):

$$\psi_\varphi : M_\varphi \to G \text{ with } \ker_\psi \stackrel{\text{def}}{=} \ker(M_\varphi \to G) = H_1(X).$$

Given $(G, \mathbf{C})$, finding fulfilling $X \Leftrightarrow$ Nonempty *Nielsen classes*:

$$\mathrm{Ni}(G, \mathbf{C}) = \{\boldsymbol{g} \in \mathbf{C} | \langle \boldsymbol{g} \rangle = G, g_1 \cdots g_r = 1\}.$$

Note: Product-one condition. $g_1 \cdots g_r = 1$.

Part I: Finite Group Property Gives Tower of Spaces

D_p (order $2p$, p odd) fact: With C_2, conjugacy class of involution, if $(g_1, g_2) \in \mathrm{C}_{2^2}$ generate D_p, then for $\boldsymbol{g}' \in (D_{p^{k+1}})^2 \cap \mathrm{C}_{2^2}$ over $\boldsymbol{g}$, $\langle \boldsymbol{g}' \rangle = D_{p^{k+1}}$, $k \geq 1$.

For $k \geq 0 \; \exists \; \psi_k : G_{k,\mathrm{ab}} \stackrel{\mathrm{def}}{=} G_k \longrightarrow G_0 = G \longrightarrow 1$ with kernel $(\mathbb{Z}/p^k)^u$, $u \geq 1$ (if $p||G|$: independent of k), with $G_k \leftrightarrow G_0$ as $D_{p^{k+1}} \leftrightarrow D_p$. Key: G_k is *versal* for abelian exponent p^k extensions of G_0.

Note: $u = 1$ if and only if G is p-supersolvable (slight generalization of dihedral groups).

$(G,\mathbf{C},p)$ Fact, with $\mathbf{C}$ p', produces spaces

(*) For $\boldsymbol{g}\in\mathbf{C}$ with $\langle\boldsymbol{g}\rangle=G$, each $\boldsymbol{g}'\in G_k^r\cap\mathbf{C}$ over $\boldsymbol{g}$ has $\langle\boldsymbol{g}'\rangle=G_k$.

(*) requires G p-perfect: no $G\to\mathbb{Z}/p\to 1$.

Operations on Nielsen classes generate H_r:

sh : $(g_1,\dots,g_r)\mapsto(g_2,\dots,g_r,g_1)$

$q_2:(g_1,\dots,g_r)\mapsto(g_1,g_2g_3g_2^{-1},g_2,g_4,\dots,g_r)$.

Homological condition produces spaces:

Equivalent existence of three projective sequences:

(**) $M_\varphi\to G$ extends to $M_{\varphi,k}\to G_k$, $k\ge 1$ $[\mod H_r]$ $\Leftrightarrow\{O_k=H_r(\boldsymbol{g}_k)\}_{k=0}^\infty$ of H_r orbits on $\{\mathrm{Ni}(G_k,\mathbf{C})^{\mathrm{in}}\}_{k=0}^\infty\Leftrightarrow$ [components of] *reduced, inner Hurwitz spaces* (dim. $r-3$) $\{\mathcal{H}(O_k)^{\mathrm{in,rd}}\subset\mathcal{H}(G_k,\mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^\infty$.

Modular Curves and their generalization

Assume G centerless. $\exists$ unique versal central extension $\mu_{G,p} : R_p \to G$ with $\ker(R_p \to G)$: p part of Schur multiplier of G.

[F(ried)-W(eigel) [Lum, Cor. 4.19], [We]] (**) holds $\Leftrightarrow M_\varphi \to G$ extends to $M_\varphi \to R_p$.

Modular Curve Fact: Modular Curve sequence $\{X_1(p^{k+1})\}$ automatic from $\mathrm{Ni}(D_p, \mathbf{C}_{2^4}, p)$ by compactifying the Hurwitz spaces. Schur multiplier of D_p is trivial $\implies$ [F-W] hypothesis.

Def: M(odular)T(ower): (Nonempty) Projective system of $H_r = \langle q_2, \mathbf{sh}\rangle$ orbits on $\{\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$.

Part II: Odd Pure-Cycle Modular Towers

- $g \in S_n$ is *pure-cycle* if exactly one cycle has length > 1.
- Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ is *pure-cycle* if all conjugacy classes are pure-cycle (of length $\{d_1, \ldots, d_r\} = \boldsymbol{d}$): $\mathbf{C} = \mathbf{C}_{\boldsymbol{d}}$.

Assume $G \le S_n$ transitive and $\mathbf{C}^{S_n} \overset{\text{def}}{=} \mathbf{C}_{d_1 \cdots d_r}$ image of $\mathbf{C}$ in S_n, with d_i s all odd.

Iff Genus condition for $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{g}}) \ne \emptyset$:

$$\mathbf{g}_{\boldsymbol{d}} = \mathbf{g}_{d_1 \cdots d_r} \overset{\text{def}}{=} \frac{\sum_{i=1}^{r} d-1}{2} - (n-1) \text{ is non-negative.}$$

[Wilson]: For $\boldsymbol{g}$ odd, $r \ge 3$ and $\mathbf{g}_{\boldsymbol{d}} = 0$, $G = A_n$.
[LOs06]: One H_r orbit.

F(ried)-S(erre): MTs over $\mathcal{H}(A_n, \mathbf{C}_{\boldsymbol{d}})$ ([LUM,§1], [Ser90a])

$\mathrm{Spin}_n^+ \to O_n^+$: nonsplit connected degree 2 cover of O_n^+.

$\mathrm{Spin}_n =$ pullback of A_n to Spin_n^+:

$\ker(\mathrm{Spin}_n \to A_n) = \{\pm 1\}$ is Schur multiplier of A_n, $n \ge 4$.

Odd order $g \in A_n$ has a unique odd order lift, $\hat{g} \in \mathrm{Spin}_n$.

Let $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{\boldsymbol{d}})$. *Small* lifting invariant:

$s(\boldsymbol{g}) = s_{\mathrm{Spin}_n}(\boldsymbol{g}) = \hat{g}_1 \cdots \hat{g}_r \in \{\pm 1\}$.

Theorem 1. $\exists$ *at least one* MT *over a component* $\leftrightarrow H_r$ *orbit* O *on* $\mathrm{Ni}(A_n, \mathbf{C}_{\boldsymbol{d}}) \Leftrightarrow$ *something in* $\mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}_{\boldsymbol{d}})$ *over* O.

If $\mathbf{g}_{\boldsymbol{d}} = 0 \Leftrightarrow \sum_{i=1}^r \frac{d_i^2-1}{2} \equiv 0 \mod 2$.

Cusp on an H_r orbit $O \subset \mathrm{Ni}(G, \mathbf{C})$

- $r \geq 5$: An orbit of $\mathrm{Cu}_r = \langle q_2 \rangle$
- $r = 4$: An orbit of $\mathrm{Cu}_4 = \langle q_2, \mathbf{sh}^2, q_1 q_3^{-1} \rangle = \langle q_2, \mathcal{Q}'' \rangle$.

MiddleProduct: $(g_1, g_2, g_3, g_4) \mapsto \mathrm{ord}(g_2 g_3) \stackrel{\mathrm{def}}{=} (\boldsymbol{g})\mathbf{mpr}$.

- p cusp: $\mathrm{Cu}_r(\boldsymbol{g})$ for which $p^{\mu_p(\boldsymbol{g})} || (\boldsymbol{g})\mathbf{mpr}$, $\mu_p(\boldsymbol{g}) > 0$.
- g(roup)-p': $U_{1,4}(\boldsymbol{g}) = \langle g_1, g_4 \rangle$, $U_{2,3}(\boldsymbol{g}) = \langle g_2, g_3 \rangle$ are p' groups.

Example 2 (Two cusps on $X_0(p)$). 1st: (p cusp) is Cu_4 orbit of H(arbater)-M(umford) rep. $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$, g_1, g_2 distinct involutions in D_p. 2nd: (width 1, special g-p') is orbit of $(\boldsymbol{g})\mathbf{sh} = (g_1^{-1}, g_2, g_2^{-1}, g_1)$.

Part III: Given a MT, $\{O_k \subset \text{Ni}(G_k, \mathbf{C})^{\text{in}}\}_{k=0}^{\infty}$, classify when there is a p cusp on O_k for $k >> 0$.

Main Conj.: K a number field, then $\mathcal{H}'_k(K) = \emptyset$ for $k >> 0$. For $r = 4$, this holds if $g'_k > 0$ and there is a p-cusp.

$H_4/\mathcal{Q}'' \stackrel{\text{def}}{=} \bar{M}_4 = \langle \gamma_0, \gamma_1, \gamma_\infty \rangle, q_1 q_2 \mapsto \gamma_0$ (order 3),
shift $= q_1 q_2 q_3 \mapsto \gamma_1$ (order 2),
$q_2 \mapsto \gamma_\infty$ ($j = \infty$ monodromy generator),
satisfying the product-one relation: $\gamma_0 \gamma_1 \gamma_\infty = 1$.

Tower levels are upper half-plane quotients, j-line covers: γ_is acting on $\text{Ni}(G, \mathbf{C})^{\text{in},rd} \stackrel{\text{def}}{=} \text{Ni}(G, \mathbf{C})^{\text{in}}/\mathcal{Q}''$ are their branch cycles.

Cusp rep listings, $r = 4$, $\mathrm{Ni}^{\text{abs or in}}_{(\frac{n+1}{2})^4}$, all d_i s equal

Define $x_{i,j} = (i\; i{+}1 \cdots j)$. g-2$'$ cusps here are shifts of HM reps. $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Mod conjugation by A_n they are

$$\mathrm{HM}_1 \stackrel{\text{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

$$\mathrm{HM}_2 = (\mathrm{HM}_1)q_1 \stackrel{\text{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

Proposition 3. *For $n \equiv 5 \mod 8$, HM_1 and HM_2 are not inner equivalent $\implies$ one braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$: One component defined/$\mathbb{Q}$.*

For $n \equiv 1 \mod 8$, if $h \in S_n \setminus A_n$, there is no braid between $\boldsymbol{g}$ and $h\boldsymbol{g}h^{-1} \implies$ two braid orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$: Two components conjugate/$\mathbb{Q}(\sqrt{-\frac{n+1}{2}})$.

$\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ Table of Cusp reps.(row starts $\mathrm{ord}(g_2g_3)$): **sh** applied to $\mathrm{Cu}_4(\mathrm{HM}_1) =$

$$\{\mathrm{HM}_{1,t} = (x_{\frac{n+1}{2},1}, x_{1+t,\frac{n+1}{2}+t}, x_{\frac{n+1}{2}+t,n+t}, x_{n,\frac{n+1}{2}})\}_{t=0}^{n-1}.$$

1: $(\mathrm{HM}_{1,0})\mathbf{sh} = (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

3: $(\mathrm{HM}_{1,1})\mathbf{sh} = (x_{2,\frac{n+3}{2}}, (\frac{n+3}{2}\ \dots\ n\,1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

5: $(\mathrm{HM}_{1,2})\mathbf{sh} = (x_{3,\frac{n+5}{2}}, (\frac{n+5}{2}\ \dots\ n\,1\,2), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

$\dots$

n: $(\mathrm{HM}_{1,\frac{n-1}{2}})\mathbf{sh} = (x_{\frac{n+1}{2}\,n}, (n\,1\ \dots\ \frac{n-1}{2}), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

For cusps of $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, $n \equiv 5 \mod 8$: Two each of width k for each odd $3 \le k \le n$, $O'_{k;j}$, $j = 1, 2$, one, $O_{1,2}$, of width 2 (shift of H-M cusp). None are 2 cusps.

sh-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{3^4}^{\mathrm{in,rd}}$

sh-incidence pairing on Cu_4 orbits $\mod \mathcal{Q}''$:

$(O, O') \mapsto |O \cap (O')\mathbf{sh}|$: $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$

Orbit	$O'_{5;1}$	$O'_{5;2}$	$O'_{3;1}$	$O'_{3;2}$	$O_{1,2}$
$O'_{5;1}$	0	2	1	1	1
$O'_{5;2}$	2	0	1	1	1
$O'_{3;1}$	1	1	0	1	0
$O'_{3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

Lemma 4. *Fixed points of γ_0 or γ_1 contribute to diagonal of* **sh***-incidence matrix.* $\mathbf{g}_{3^4} = 0$:

$$2(18 + \mathbf{g}_{3^4} - 1) = 2 \cdot 18/3 + 18/2 + (1 + 2 \cdot 2 + 2 \cdot 4).$$

2 cusps in Liu-Osserman cases

List 3-tuples $(g_2, g_3, (g_2g_3)^{-1})$ for each $O'_{2u+1;j}$, $3 \le u \le \frac{n-1}{2}$, $j = 1, 2$: $\mathrm{ord}(g_2g_3) = 2u+1$; $\langle g_2, g_3\rangle \sim A_{u+\frac{n+1}{2}}$. [LUM, Fratt. Princ. 3]: Level 1 has only 2 cusps above $O'_{2u+1;j}$ iff $s_{\mathrm{Spin}_n/A_n}(g_2, g_3, (g_2g_3)^{-1}) = \frac{\mathrm{ord}(g_2g_3)^2-1}{8}$ (F-S)$\equiv 1 \mod 2$.

Theorem 5. *If a cusp branch is both H-M and p, then* MT *cusp tree contains a spire: sub-tree isomorphic to a modular curve cusp tree. Holds for $p = 2$ at level 1, for* L-O $n \equiv 5 \mod 8$. *Doesn't hold for $n \equiv 1 \mod 8$.*

SPIRE: Growth of p cusps with level: Subscript is power of p dividing the middle product.

Level 1 :	$\bullet_p$			
Level 2 :	$\bullet_{p^2}$	$\bullet_p$		
Level 3 :	$\bullet_{p^3}$	$\bullet_{p^2}$	$\bullet_p$	
$\cdots$:	$\ldots$	$\ldots$	$\ldots$	$\ldots$

App. A: Atomic Orbital type; and 2 cusp comment

Correspondence with atomic orbitals: $n \leftrightarrow$ orbital energy level, for each n, total inner reduced Nielsen classes: $2\cdot(\sum_{\text{odd } k=0}^{n} k = 2\cdot n^2)$.

2 cusps for L-O $n = 9$, $\mathbf{C}_{5^4}$: $\ell \in \{1,3,5,7,9\}$ (each component has such width cusps): two 2 cusps ($\Leftrightarrow \ell = 3,5$) at level 1 for certain. Above cusps with middle products 7 and 9, not clear there is a 2 cusp on every component.

Abbreviated References: [LUM] has much more

[BFr02] P. Bailey and M. D. Fried, *Hurwitz monodromy, spin separation and higher levels of a Modular Tower*, in Proceed. of Symposia in Pure Math. **70** (2002) editors M. Fried and Y. Ihara, 1999 von Neumann Symposium, August 16-27, 1999 MSRI, 79–221.

[STMT] A. Cadoret, *Modular Towers and Torsion on Abelian Varieties*, preprint May, 2006.

[D06] P. Dèbes, *Modular Towers: Construction and Diophantine Questions*, same vol. as [LUM].

[Def-Lst] Select from the list in www.math.uci.edu/conffiles_rims/deflist-mt/full-deflist-mt.html of present MT-related definitions. 09/05/06 examples: Branch-Cycle-Lem CFPV-Thm Cusp-Comp-Tree FS-Lift-Inv Hurwitz-Spaces Main-MT-Conj Modular-Towers Nielsen-Classes RIGP Strong-Tors-Conj mt-rigp-stc p-Poincare-Dual sh-Inc-Mat. A similar URL, www.math.uci.edu/conffiles_rims/deflist-mt/full-paplist-mt.html, is a repository for not just mine, but also of the growing list of those joining the MT project.

[FrK97] M. Fried and Y. Kopeliovic, *Applying Modular Towers to the inverse Galois problem*, Geometric Galois Actions II Dessins d'Enfants, Mapping Class Groups . . . , vol. 243, Cambridge U. Press, 1997, London Math. Soc. Lecture Notes, 172–197.

[Fr77] M. Fried, *Fields of definition of function fields and Hurwitz families and groups as Galois groups*, Communications in Algebra **5** (1977), 17–82.

[FrV] Michael D. Fried and Helmut Völklein, *The inverse Galois problem and rational points on moduli spaces*, Math. Ann. **290** (1991), no. 4, 771–800.

[LUM] M. D. Fried, *The Main Conjecture of Modular Towers and its higher rank generalization*, in Groupes de Galois arithmetiques et differentiels (Luminy 2004; eds. D. Bertrand and P. Dèbes), Seminaires et Congres, **13** (2006), 165–230.

[AGLI] M. D. Fried, Alternating groups and lifting invariants, (2006), 1–36.

[LOs06]F. Liu and B. Osserman, *The Irreducibility of Certain Pure-cycle Hurwitz Spaces*, preprint as of August 10, 2006.

[Ser90a]J.-P. Serre, *Relêvements dans* $\tilde{A}_n$, C. R. Acad. Sci. Paris **311** (1990), 477–482.

[We]T. Weigel, *Maximal p-frattini quotients of p-poincare duality groups of dimension 2*, volume for O.H. Kegel on his 70th birthday, Arkiv der Mathematik–Basel, 2005.