Maximal Frattini quotients of p-Poincaré Mapping class groups

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http://math.uci.edu/~mfried → §1.a →

Generalizing modular curve properties to Modular Towers

→ #1 mt-overview.html

Let $\varphi : X \to \mathbb{P}^1_z$ be a function on a Riemann surface X, and assume φ is Galois, with group G.

Then, φ defines these quantities:

- Unordered branch points $\boldsymbol{z} = \{z_1, \dots, z_r\} \in U_r$;
- Conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$ in G; and
- A *Poincaré extension* of groups:

$$\psi_\varphi : M_\varphi \to G \text{ with } \ker_\psi \stackrel{\text{def}}{=} \ker(M_\varphi \to G) = \pi_1(X).$$

Using Classical Generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$

Denote $(\mathcal{P}_1, \dots, \mathcal{P}_r)$ (see App. A) $\mapsto$ an isotopy class of r generators $\bar{\boldsymbol{g}} = (\bar{g}_1, \dots, \bar{g}_r)$.

Refer to their images in M_φ also as $\bar{\boldsymbol{g}}$, and their images in G by $(g_1, \dots, g_r) = \boldsymbol{g}$.

Then, $\boldsymbol{g}$ is in the Nielsen class of $(G, \mathbf{C})$:

$$\mathrm{Ni}(G, \mathbf{C}) \stackrel{\mathrm{def}}{=} \{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \Pi(\boldsymbol{g}) \stackrel{\mathrm{def}}{=} g_1 \cdots g_r = 1\}.$$

Notion: Given classical generators $\bar{\boldsymbol{g}}$ of M_φ, rename it $M_{\bar{\boldsymbol{g}}}$: ψ_φ becomes $\psi_{\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \longrightarrow G$, by $\bar{g}_i \mapsto g_i$.

E(xtension) P(roblem) (Item #1 below)

Given $(\boldsymbol{g}, \mathbf{C})$, and a prime p:

1. When does $\psi_{\boldsymbol{g}}$ extend to all $H \to G \to 1$ with p-group kernel? Abelianized version (App. C): To all H with $\ker(H \to G)$ abelian.
2. How does this depend on $\boldsymbol{g}$?
3. What equivalence relation on extensions gives a reasonable description of all cases?
4. Why should this concern mathematics?

Non-obvious Reductions

- Complete $M_{\bar{\boldsymbol{g}}}$ so $\ker_{\psi_\varphi}=$ pro-p completion of $\pi_1(X)$.
- Restrict in #1 (p. 3) to p-Frattini covers of G.
- Any $g \in$ **C** must have order prime to p.
- G is p-perfect (no $\mathbb{Z}/p$ quotient; or #1 impossible).

Equivalent: When are all p-Frattini covers $H \to G \to 1$ achieved by unramified extensions $Y_H \to X$?

Deformation equivalence of extensions

If φ were a cyclic cover of $\mathbb{P}^1$, we could write it by hand. It isn't. Further, why deal one cover at-a-time? Consider all covers with $(G, \mathbf{C})$ as their data: In the Nielsen class.

Deformation Conclusion: Can always start by fixing branch points $\boldsymbol{z}^0$. Any cover (with branch points $\boldsymbol{z}$) deforms to a cover with branch points $\boldsymbol{z}^0$. Then, $M_{\bar{\boldsymbol{g}}}$ and any of its extension properties deform with it.

Braid equivalence of extensions

The *Hurwitz Monodromy group* H_r has two generators given by their action on $\bar{\boldsymbol{g}}$:

- Shift: **sh** : $\bar{\boldsymbol{g}} \mapsto (\bar{g}_2, \dots, \bar{g}_r, \bar{g}_1)$; and
- 2ndTwist: $q_2 : \bar{\boldsymbol{g}} \mapsto (\bar{g}_1, \bar{g}_2\bar{g}_3\bar{g}_2^{-1}, \bar{g}_2, \bar{g}_4, \dots)$: $q_{i+2} \stackrel{\text{def}}{=} \mathbf{sh}^i q_2 \mathbf{sh}^{-i}$.

Braid Comments: H_r is automorphism group of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$ preserving classical generators. It acts compatibly on these:

- Inner Nielsen Classes: $\mathrm{Ni}(G, \mathbf{C})/G \stackrel{\text{def}}{=} \mathrm{Ni}^{\text{in}}$
- Absolute Nielsen classes: $\mathrm{Ni}(G, \mathbf{C})/N_{S_n}(G) \stackrel{\text{def}}{=} \mathrm{Ni}^{\text{abs}}$ (given $G \le S_n$ a permutation representation)
- Poincaré extensions: $\psi_{\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$, *preserving* extension properties of $\psi_{\boldsymbol{g}}$

Modular Towers and Start of Criterion for GOAL 1

GOAL 1: Given $(G, \mathbf{C}, p)$, understand projective systems of H_r orbits acting on $\{\mathrm{Ni}(H, \mathbf{C})^{\mathrm{in}}\}_{H \to G}$: Running over p-Frattini covers $H \to G$.

Reduction: Take $G_1 \to G = G_0$ to be the maximal p-Frattini cover of G with elementary p group kernel. Let $G_{k+1} = G_1(G_k)$. In GOAL 1 need only the case H runs over the G_k s.

Def: M(odular) T(ower): A projective system $\{O_k = H_r(\boldsymbol{g}_k)\}_{k=0}^{\infty}$ of H_r orbits on $\{\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$.

Inductive Existence of a MT: [Lum, Cor. 4.19], [We]

Let $\mu_k : R_k \to G_k$ be the universal exponent p central extension of G_k:

- $G_{k+1} \to G_k$ factors through μ_k.
- $\ker(R_k \to G_k) =$ max. elementary p-quotient of G_k s Schur multiplier.

Proposition 1 (App. C– Abel. Vers.). *If p-perfect G has no p-center, then neither does G_k, $k \geq 1$.*

$H_r(\boldsymbol{g}_k) \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}$ is in the image of $\mathrm{Ni}(G_{k+1}, \mathbf{C})^{\mathrm{in}} \Leftrightarrow \boldsymbol{g}_k$ is in the image of $\mathrm{Ni}(R_k, \mathbf{C}) \Leftrightarrow H_r$ orbit of $M_{\boldsymbol{g}} \to G$ extends through all G_k.

Cusps and the p Cusp Problem

Cusp on an H_r orbit $O \subset \mathrm{Ni}(G, \mathbf{C})$:

- $r \geq 5$: Orbit of $\mathrm{Cu}_r = \langle q_2 \rangle$
- $r = 4$: $\mathrm{Cu}_r = \langle q_2, \mathbf{sh}^2, q_1 q_3^{-1} \rangle$.

Essential data from conjugacy class of Cu_r, so can substitute q_i for q_2.

p cusp: represented by $\boldsymbol{g} \in O$ for which $p^{\mu_p(\boldsymbol{g})} || \mathrm{ord}(g_2 g_3) \stackrel{\mathrm{def}}{=} (\boldsymbol{g})\mathbf{mpr}, \mu_p(\boldsymbol{g}) > 0$ (p-mult. of $\boldsymbol{g}$).

Other cusp types for $r = 4$ (App. B for $r > 4$)

- g(roup)-p': $U_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle$ and $U_{2,3}(\boldsymbol{g}) = \langle g_2, g_3\rangle$ are p' groups
- o(nly)-p': Not a p cusp, but $U_{1,4}(\boldsymbol{g})$ or $U_{2,3}(\boldsymbol{g})$ not p'.

GOAL 2: Given a MT, $\{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, classify when there is a p cusp on O_k for $k >> 0$.

Proposition 2 (g-p' MT). *If O_0 has a g-p' cusp, then a MT, $\mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, lies over it.*

MT Geometric correspondence

$\mathcal{O} = \{O_k \subset \text{Ni}(G_k, \mathbf{C})^{\text{in}}\}_{k=0}^{\infty} \Leftrightarrow \{\mathcal{H}'_k\}_{k=0}^{\infty}$ where:

- $\mathcal{H}'_k$s are (normal) absolutely irreducible algebraic varieties (dim$=r-3$);
- Cusps at level k correspond to divisors on the normal compactification $\bar{\mathcal{H}}'_k$.
- ${}_0\boldsymbol{g} \in O_0$ a p cusp $\implies \mu_p({}_k\boldsymbol{g}) = k + \mu_p({}_0\boldsymbol{g})$.
- $r = 4$: $\mathcal{H}'_k$, upper-half plane quotient, j-line cover, ramific. order dividing 3 (resp. 2) over 0 (resp. 1).

p Cusps and Main MT Conjecture

Main Conj.: K a number field, then $\mathcal{H}'_k(K) = \emptyset$ for $k >> 0$. For $r = 4$, let g'_k be the genus of $\bar{\mathcal{H}}_k$.

Proposition 3. *If $g'_0 > 0$, (resp. $= 0$) and, for some k, $\mathcal{H}'_k$ has a p cusp (resp. three p cusps), then Main Conj. holds for $\mathcal{O}$.*

Example 4 (Liu-Osserman). Among pure-cycle classes, $n \equiv 1 \mod 4$, $\mathbf{C}_{(\frac{n+1}{2})^4}$, four reps of the $\frac{n+1}{2}$-cycle conjugacy class is, maybe, the hardest case. Here: $G_0 = A_n$.
For $p = 2$ (resp.> 2), $R_0 = \mathrm{Spin}_n$ (resp. $R_0 = A_n$).

Cusp rep listings, $r = 4$, $\mathrm{Ni}^{\text{abs or in}}_{(\frac{n+1}{2})^4}$

Define $x_{i,j} = (i\, i{+}1 \cdots j)$. g-2′ cusps here are shifts of HM reps. $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Mod conjugation by A_n they are

$$\mathrm{HM}_1 \stackrel{\text{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

$$\mathrm{HM}_2 \stackrel{\text{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

Proposition 5. *For $n \equiv 1 \mod 8$ (resp. $n \equiv 5 \mod 8$), HM_1 and HM_2 are (resp. are not) inner equivalent. For the latter, $(\mathrm{HM}_1)q_1 = \mathrm{HM}_2$ implies there is a braid between $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$ and $h\boldsymbol{g}h^{-1}$ for any $h \in S_n$, and so there is one braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.*

For $n \equiv 1 \mod 8$, if $h \in S_n \setminus A_n$, there is no braid between $\boldsymbol{g}$ and $h\boldsymbol{g}h^{-1}$. Exactly two braid orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

Cusp rep listings, $r=4$, $\mathrm{Ni}_{(\frac{n+1}{2})^4}$: **Table**$_{(\mathrm{HM}_{1,0})\mathrm{Cu}_4\circ\mathbf{sh}}$

$\mathrm{Cu}_4(\mathrm{HM}_1)$: $\{\mathrm{HM}_{1,t}=(x_{\frac{n+1}{2},1},x_{1+t,\frac{n+1}{2}+t},x_{\frac{n+1}{2}+t,n+t},x_{n,\frac{n+1}{2}})\}_{t=0}^{n-1}$ (subscripts $\mod n$). Row starts $\mathrm{ord}(g_2g_3)$:

1: $(\mathrm{HM}_{1,0})\mathbf{sh}=(x_{1,\frac{n+1}{2}},x_{\frac{n+1}{2},n},x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

3: $(\mathrm{HM}_{1,1})\mathbf{sh}=(x_{2,\frac{n+3}{2}},(\frac{n+3}{2}\ \dots\ n\,1),x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

5: $(\mathrm{HM}_{1,2})\mathbf{sh}=(x_{3,\frac{n+5}{2}},(\frac{n+5}{2}\ \dots\ n\,1\,2),x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

$\dots$

n: $(\mathrm{HM}_{1,\frac{n-1}{2}})\mathbf{sh}=(x_{\frac{n+1}{2}n},(n\,1\ \dots\ \frac{n-1}{2}),x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

n: $(\mathrm{HM}_{1,\frac{n+1}{2}})\mathbf{sh}=((\frac{n+3}{2}\ \dots\ n\,1),x_{1,\frac{n+1}{2}},x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

$\dots$

5: $(\mathrm{HM}_{1,n-2})\mathbf{sh}=((n{-}1\,n\,1\ \dots\ \frac{n-3}{2}),x_{\frac{n-3}{2},n-2},x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

3: $(\mathrm{HM}_{1,n-1})\mathbf{sh}=((n\,1\ \dots\ \frac{n-1}{2}),x_{\frac{n-1}{2},n-1},x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1})$

sh-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{34}^{\mathrm{in,rd}}$

Pairing on Cu_4 orbits: $(O, O') \mapsto |O \cap (O')\mathbf{sh}|$. $O_{5,5;2}$ (resp. $O_{1,2}$) indicates 2nd **mpr** 5, width 5 (resp. only **mpr** 1, width 2) orbit. **sh**-incidence gives $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ has genus 0.

All $\mathcal{H}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$ for $n \equiv 1 \mod 4$ have genus 0.

Orbit	$O_{5,5;1}$	$O_{5,5;2}$	$O_{3,3;1}$	$O_{3,3;2}$	$O_{1,2}$
$O_{5,5;1}$	0	2	1	1	1
$O_{5,5;2}$	2	0	1	1	1
$O_{3,3;1}$	1	1	0	1	0
$O_{3,3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

Complete orbit for $\bar{M}_4 = \langle \mathbf{sh}, \gamma_\infty \rangle$ on $\mathrm{Ni}_{34}^{\mathrm{in,rd}}$ in 2-steps: Apply $(\mathbf{sh} \circ \mathrm{Cu}_4)^2$ to H-M rep.

Fried-Serre Lifting Invariant formula

Level 0 Dilemma: $\mathrm{ord}(g_2g_3) = \ell$ in each line of **Table**$_{(\mathrm{HM}_{1,0})\mathrm{Cu}_4\circ\textbf{sh}}$ is an odd: no 2 cusps at level 0.

Help by Level 1: The exact condition for each cusp at level 1 above the cusp in **Table**$_{(\mathrm{HM}_{1,0})\mathrm{Cu}_4\circ\textbf{sh}}$ to be a 2 cusp is that $\frac{\ell^2-1}{8} \equiv 1 \mod 2$.

Main Conjecture?: $n = 5$, $\ell \in \{1, 3, 3, 5, 5\}$, so four ($> 3$) 2 cusps ($\frac{3^2-1}{2} \equiv \frac{5^2-1}{2} \equiv 1 \mod 2$).

$n = 9$, $\ell \in \{1, 3, 5, 7, 9\}$: two 2 cusps ($\Leftrightarrow \ell = 3, 5$) at level 1 for certain, but above cusps with middle products 7 and 9, not clear there is a 2 cusp. Need more info on level 1 cusps.

All $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ satisfy Main Conjecture for $p = 2$

Get all odd $1, \ldots, n$ on left side of **Table**$_{(\mathrm{HM}_{1,0})\mathrm{Cu}_4 \circ \mathbf{sh}}$, so number of 2 cusps groups with n. For $n = 17$ get ≥ 2 more: $\frac{11^2-1}{8} \equiv \frac{13^2-1}{8} \equiv 1 \mod 2$.

Appendix A: Classical Generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$

Appendix B: Classification of cusps on a MT

Appendix C: Abelianized versions of Goals 1 and 2

Item $\#1$ on p. 3 has an abelianized version $\Leftrightarrow$ Abelianized version of p. 4 Goal: When are all p-Frattini covers $H \to G \to 1$ achieved by abelian unramified extensions $Y_H \to X$? With G_k the characteristic p-Frattini cover G_k, and $\ker_k^* = (\ker(G_k \to G), \ker(G_k \to G))$, form $G_k^{\rm ab} = G_k/\ker_k^*$.

Proposition 6 (Abel. Vers. of Prop. 1, p.3). *Let $R_0' \to G$ be the maximal central p-Frattini extension of G.*

Then, for all k, $H_r(\boldsymbol{g}_0) \subset \mathrm{Ni}(G_0, \mathbf{C})^{\rm in}$ is in the image of $\mathrm{Ni}(G_k^{\rm ab}, \mathbf{C})^{\rm in} \Leftrightarrow \boldsymbol{g}_0$ is in the image of $\mathrm{Ni}(R_0', \mathbf{C})$.

Abbreviated References: [Lum] has much more

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[Def-Lst] Select from the list in www.math.uci.edu/conffiles_rims/deflist-mt/full-deflist- mt.html of present MT-related definitions. 09/05/06 examples: Branch-Cycle-Lem CFPV-Thm Cusp-Comp-Tree FS-Lift-Inv Hurwitz-Spaces Main-MT-Conj Modular-Towers Nielsen- Classes RIGP Strong-Tors-Conj mt-rigp-stc p-Poincare-Dual sh-Inc-Mat. A similar URL, www.math.uci.edu/conffiles_rims/deflist-mt/full-paplist-mt.html, is a repository for not just mine, but also of the growing list of those joining the MT project.

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