Dihedral Groups: **M**(odular) **T**(ower) view of modular curve towers and their cusps

Mike Fried, UC Irvine: Versions from London, Ont. Oct. 2005, Istanbul, 06/17/08. References all talks: www.math.uci.edu/ mfried/talklist-mt/ucicoll05-22-08.pdf.

- §I. Abel and Dihedral functions: Explains that levels of a MT actually first appear in Calculus.

- §II. MT view of Modular curves: Modular curves systematically use cusps. We translate those cusps into MT language to show how modular curve questions and applications generalize to MTs.

§I. Abel and Dihedral functions

1st year calculus uses $T_p(\cos(\theta)) = \cos(p\theta)$, with $T_p(w) = z$ the pth *Chebychev* polynomial to express $\cos(\theta)^p$ as a sum of $\cos(k\theta)$ terms, $0 \le k \le p$.

Consider $T_p : \mathbb{P}^1_w = \mathbb{C}_w \cup \{\infty\} \to \mathbb{P}^1_z = \mathbb{C}_z \cup \{\infty\}$ as a map between complex spheres, branched over $\{z_1, z_2, z_3\} = \{-1, +1, \infty\}$.

The trick: Induct on p to find $T^*_p(w) = 2T_p(w/2)$ so $T^*_p(x+1/x) = x^p+1/x^p$.
Then substitute $x \mapsto e^{2\pi i\theta}$.

§I.A. The dihedral group with observations

T_p (p odd) is a *dihedral* function: Attached to it is a *b(ranch) c(ycle) d(escription)*:

$$\begin{aligned} g_1 &= (2\,p)(3\,p{-}1)\cdots(\tfrac{p+1}{2}\,\tfrac{p+3}{2}) \\ g_2 &= (1\,p)(2\,p{-}1)\cdots(\tfrac{p-1}{2}\,\tfrac{p+3}{2}) \\ g_\infty &= (p\,p{-}1\,\cdots\,1) \end{aligned}$$

- (generation) $\langle g_1, g_2\rangle = D_p = \{\begin{pmatrix} \pm1 & b \\ 0 & 1\end{pmatrix}\}_{b\in\mathbb{Z}/p}$, order $2p$ dihedral group; C_2, involution conjugacy class $\leftrightarrow -1$ in upper left matrix corner.
- (conjugacy classes) g_i, $i=1,2$, are in C_2.
- (product-one) $g_1 g_2 g_\infty = 1$.

Two dihedral function questions

- Q_1: From whence the (g_1, g_2, g_∞)?
  Answer: Use App. A_1: With $r = 3$.

- Q_2: If another function $f : \mathbb{P}^1_w \longrightarrow \mathbb{P}^1_z$ with similar bcd, how related to T_p?
  Answer: $\exists$ Möbius transforms. $\alpha_1, \alpha_2 \in \mathrm{PGL}_2(\mathbb{C})$ with $f = \alpha_2 \circ T_p \circ \alpha_1^{-1}(w)$: $f \sim^{\text{Möbius}} T_p$.

  Historical generalization: Abel used more general dihedral Möbius classes.

§I.B. $r = 4$ (not 3) branch dihedral functions

Denote distinct elements of $(\mathbb{P}^1_z)^4$ by U^4. S_4 (symmetries on $\{1,2,3,4\}$) permutes coordinates of U^4.

Instead of (g_1, g_2, g_∞) take $\sim^{\text{Möbius}}$ classes of functions with bcds $(g_1, g_1^{-1}, g_2, g_2^{-1}) \in \mathbf{C}_{2^4}$, H(arbater)-M(umford) —or any 4-tuple in $\mathbf{C}_{2^4}$ with generation and product-one; the set $\mathrm{Ni}(D_p, \mathbf{C}_{2^4})$ of Nielsen classes —branched over any

$$\boldsymbol{z} = \{z_1, \ldots, z_4\} \in U^4/S_4 \stackrel{\text{def}}{=} U_4 \stackrel{\text{def}}{=} U_{4,z}.$$

Why such $f : \mathbb{P}^1_w \longrightarrow \mathbb{P}^1_z$ exists

- App. A_1 gives compact surface cover $f : X \longrightarrow \mathbb{P}^1_z$.
- Then apply R-H (App. B_2) to see X has genus 0.
- Then apply R(iemann)-R(och) to conclude X is analytically isomorphic to $\mathbb{P}^1_w$.
- Each z_i has a unique unramified $w_i \mapsto z_i$ (w_i corresponds to length 1 disjoint cycle in g_i):

$$\begin{aligned} f \leftrightarrow (\boldsymbol{w}, \boldsymbol{z}) \in U_{4,w} \times U_{4,z} &\mapsto [\sim^{\text{Möbius}}] \\ \mathrm{PGL}_2(\mathbb{C}) \backslash U_{4,w} \times U_{4,z} / \mathrm{PGL}_2(\mathbb{C}) &= (\mathbb{P}^1_{j_w} \setminus \infty) \times (\mathbb{P}^1_{j_z} \setminus \infty). \end{aligned}$$

§I.C. Dragging a function by its branch points

Continuity on the space of such f s: You can drag the classical generators on $U_{\boldsymbol{z}^0} = \mathbb{P}^1_z \setminus \boldsymbol{z}^0$ along any path $P(t), t \in [0,1]$ based at $\boldsymbol{z}^0 \in U_r$ to classical generators on $U_{P(t)}$.

Upshot: You can drag f_0 to f_t *by its branch points*. If P is closed, representing $[P] \in \pi_1(U_r, \boldsymbol{z}^0)$, then f_1 (usually) has a different bcd, denoted $(\boldsymbol{g})q_{[P]}$, (relative to the original classical generators).

Facts about $q_{[P]}$, for any $r \geq 4$

- $(\boldsymbol{g})q_{[P]}$ is an r-tuple of words in the entries of $\boldsymbol{g}$.

- For $i = 1, \ldots, r-1$, there is a $[P_i]$ so that,

$$q_i \stackrel{\text{def}}{=} q_{[P_i]} : (g_1, \ldots, g_r)$$
$$\mapsto (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_r).$$

Add the *shift*: $sh : (g_1, \ldots, g_r) \mapsto (g_2, \ldots, g_r, g_1)$.

Two generatored Hurwitz monodromy H_r: $\pi_1(U_r, \boldsymbol{z}^0) = \langle q_2, \textbf{sh} \rangle$

(For $r = 4$ has normal subgroup, $\mathcal{Q}'' = \langle q_1 q_3^{-1}, \textbf{sh}^2 \rangle$.)

§I.D. $\mathrm{PGL}_2(\mathbb{C})$ action; mapping class group $\bar{M}_r$

Denote the universal cover of a space X by $\tilde{X}$. Let $V_r \longrightarrow U_r$ be the fibration with fiber $U_{\boldsymbol{z}}$ over $\boldsymbol{z} \in U_r$.

Now add $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ action, starting from

$$(z_1, \dots, z_r) \in U^r \mapsto ((z_1)\alpha, \dots, (z_r)\alpha).$$

Induces Galois (group $\bar{M}_r$,r-branch point mapping class group), but ramified,

$$V_r/\mathrm{PGL}_2(\mathbb{C}) \longrightarrow U_r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\text{def}}{=} J_r \ (J_4 = \mathbb{P}^1_j \setminus \infty).$$

§II. MT view of modular curves

$\Gamma \le \mathrm{PSL}_2(\mathbb{Z}) = \langle \gamma_0, \gamma_1 \rangle$ of finite index: γ_0 and γ_1 of resp. orders 3 and 2. Act on upper half-plane $\mathbb{H}$. MTs recognizes $\mathrm{PSL}_2(\mathbb{Z})$ as $\bar{M}_4$ with generators $q_1 q_2 = \gamma_0, \textbf{sh} = \gamma_1 \mod \langle q_1 q_3^{-1}, \textbf{sh}^2 \rangle$.

- $\Gamma \mapsto X_\Gamma^0$, ramified cover of $U_j = \mathbb{P}_j^1 \setminus \{\infty\}$.
- Orbits of $\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the cosets of Γ in $\mathrm{PSL}_2(\mathbb{Z})$ correspond to the *cusps* over ∞.

II.A. Classical cusp description for $\Gamma_0(p^{k+1})$.

Count $\Gamma_0(p^{k+1}) \stackrel{\text{def}}{=} \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod p^{k+1}\}$ cusps: Select coset reps $\Rightarrow$ compute their γ_∞ orbits. This is the procedure of [Sh71, §1.6], but that has no sh-incidence cusp pairing as in App. E_1.

II.B. Dihedral Nielsen classes; q_2 action.

Assume p is odd. Order $2 \cdot p^{k+1}$ dihedral group (resp. its normalizer in $S_{p^{k+1}}$):

$$G_k = D_{p^{k+1}} \stackrel{\text{def}}{=} \{\begin{pmatrix} \pm 1 & b \\ 0 & 1 \end{pmatrix}\}_{b \in \mathbb{Z}/p^{k+1}}$$

$$\text{(resp. } N_k \stackrel{\text{def}}{=} \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\}_{a \in (\mathbb{Z}/p^{k+1})^*, b \in \mathbb{Z}/p^{k+1}}\text{)}.$$

Defining $\mathrm{Ni}(G_k, \mathbf{C}_{2^4})$

$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in N_k$ acts on $\{(b', 1) \mid b' \in \mathbb{Z}/p^{k+1}\}$:

$(b', 1) \mapsto (a \cdot b' + b, 1)$. With $\mathrm{C}_2 = \{\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}\}_{b \in \mathbb{Z}/p^{k+1}}$, use this [] notation for Nielsen class elements: $\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C}_{2^4}) \mapsto [b_1, b_2, b_3, b_4] \in (\mathbb{Z}/p^{k+1})^4$.

Absolute *Nielsen classes* $\mathrm{Ni}(G_k, \mathbf{C}_{2^4})^{\mathrm{abs}}$:
$\{\boldsymbol{g} = (g_1, \ldots, g_4) \in \mathbf{C}_{2^4} \mid$ (G) some $b_i \neq b_j \mod p$;
and (P-O) $b_1 - b_2 + b_3 - b_4 \equiv 0 \mod p^{k+1}\}/N_k$.
(for *inner* classes replace N_k by G_k).

II.C. Cusps as $\mathrm{Cu}_4 = \langle \mathcal{Q}'', q_2 \rangle$ orbits and p-cusps

G(eneration) is a Frattini property: It holds in G_0, so it holds in the Frattini cover $G_k \to G_0$ for any k.

P(roduct)-O(ne) is a translation of $g_1 g_2 g_3 g_4 = 1$ in this dihedral case. It descends from classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$ being paths whose ordered product is homotopic to one.

With $\mathcal{Q}'' = \langle q_1 q_3^{-1}, \mathbf{sh}^2 \rangle$, *cusps* $\Leftrightarrow$ Cu_4 orbits.

Trivial $\mathcal{Q}''$ action

Princ. 1: Denote the braid orbit of $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^*$ (*$=$in or abs) by $O_{\boldsymbol{g}}$. If Nielsen class of $\boldsymbol{g}$ fixed by $q_1 q_3^{-1}$ and **sh**, then $\mathcal{Q}''$ fixes class of any $\boldsymbol{g}' \in O_{\boldsymbol{g}}$.

Apply Princ. 1 to $\boldsymbol{g} = \boldsymbol{g}_{\mathrm{H-M}} \mapsto [0, 0, 1, 1]$. This is the unique absolute *Harbater-Mumford rep.*: $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$. Later see: Just one braid orbit.

Middle Product: For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, $\mathbf{mpr}_{\boldsymbol{g}} = \mathrm{ord}(g_2 g_3)$.

$\mathrm{Cu}_4(\boldsymbol{g})$ is p-cusp: p divides $\mathbf{mpr}_{\boldsymbol{g}}$.

Frattini property: Multiply by p to get width of next level cusps over p-cusps.

II.D. Cusp width principle

Princ. 2: If G centerless, either the q_2 orbit on $\boldsymbol{g}^{\mathrm{in}}$ has width $2 \cdot \mathbf{mpr}_{\boldsymbol{g}}$ or $\mathbf{mpr}_{\boldsymbol{g}}$ is odd, and

$(g_2g_3)^{\frac{\mathbf{mpr}_{\boldsymbol{g}}-1}{2}}g_2$ has order 2 mod $\mathrm{Cen}_G(\langle g_1, g_4\rangle)$.

If $\boldsymbol{g} \in \mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4}) \mapsto [b_1, \dots, b_4]$, then

$$\mathbf{mpr}_{\boldsymbol{g}} = \text{order of } b_{\boldsymbol{g}} \stackrel{\mathrm{def}}{=} b_3 - b_2 \text{ in } \mathbb{Z}/p^{k+1}.$$

Product of an odd (resp. even) number of elements from C_2 is in C_2 (resp. translat. by some $b \in \mathbb{Z}/p^{k+1}$). Princ. 2 gives this list of cusp reps.:

$$\{[b_1, b_2+mb_{\boldsymbol{g}}, b_3+mb_{\boldsymbol{g}}, b_4]\}_{m=0}^{\mathbf{mpr}_{\boldsymbol{g}}-1}.$$

II.E. Normalizations for listing all cusps

Norm$_1$: Conjugate $[b_1,\dots,b_4]$ by $\langle\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\rangle$ to assume $b_1=0$ and $b_2-b_3+b_4=0$.

Norm$_2$: For $\mathrm{Ni}^{\mathrm{abs}}$, if $b_2-b_3=ap^u$, $a\in\mathbb{Z}/p^{k+1-u}$ and $(a,p)=1$, conjugate by $\left(\begin{smallmatrix}a^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)$ so $a=1$.

Norm$_3$: Allows further conjugation by

$$H_u=\{\alpha=1+bp^{k+1-u}\in\mathbb{Z}/p^{k+1} \mod p^u, b\in\mathbb{Z}/p^u\}.$$

All are p-cusps or g-p' cusps

Take $c = b_2$, $b_3 = c - p^u$ (u is a parameter).

$u = 0$: $(b_2, b_3) = (c, c-1)$ has q_2 orbit of width p^{k+1} containing $\boldsymbol{g}_{\mathrm{H-M}} \mapsto [0,0,1,1]$.

$u > 0$: $\langle \boldsymbol{g} \rangle = D_{p^{k+1}} \implies (c, p) = 1$. Conjugate by H_u to assume $c \in \mathbb{Z}/p^{k+1-u}$ is p':

Width $= |$residues $\bmod p^{k+1-u}$ differing by multiplies of $p^u|$.

Conclude: $\varphi(p^{k+1-u})$ Nielsen classes in Cu_4 orbits of width p^{k+1-2u} (resp. 1) if $k+1-2u \ge 0$ (resp. $k+1-2u < 0$).

$u = k+1$, a g-p' cusp: $(b_2, b_3) = (1, 1)$: $(\boldsymbol{g}_{\text{H-M}})$**sh** (width 1), $H_{2,3} = \langle g_2, g_3 \rangle$, $H_{1,4} = \langle g_1, g_4 \rangle$, both p' groups (here cyclic).

II.F. MT account for one H_4 orbit

Absolute case: $(\boldsymbol{g})$**sh** $\mapsto [c, c-p^u, -p^u, 0]$. As $(c,p)=1$, from above, cusp orbit contains $[c,c,0,0]$.

sh-incidence pairing (App. E_1) has H-M cusp intersect all other cusps (matrix has *one* block).

Conclude: Ni^{abs} gives a degree $p^{k+1}+p^k$ cover $\bar{\mathcal{H}}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\text{abs,rd}} \longrightarrow \mathbb{P}^1_j$. Respective result for Ni^{in} is degree $\frac{\varphi(p^{k+1})}{2}(p^{k+1}+p^k)$.

MT account for one inner H_4 orbit

From absolute case, need only braid between inner classes of H-M reps: $\{[0,0,c,c]\}_{c\in(\mathbb{Z}/p^{k+1})^*}$.

Shift $[0,1,1+c,c]$ (in Cu_4 orbit of $[0,0,c,c]$) to get $[1,1+c,c,0]=\boldsymbol{g}'$. Then, $[1,1,0,0]$ is in cusp of $\boldsymbol{g}'$ which is (above) inner equivalent to $\boldsymbol{g}_{\text{H-M}}$.

Problem: Use App. C_1 with MT description to compute the genus of $X_0(p^{k+1})$ and $X_1(p^{k+1})$.

I.G. Summary: Modular curve vs all MT level cusps

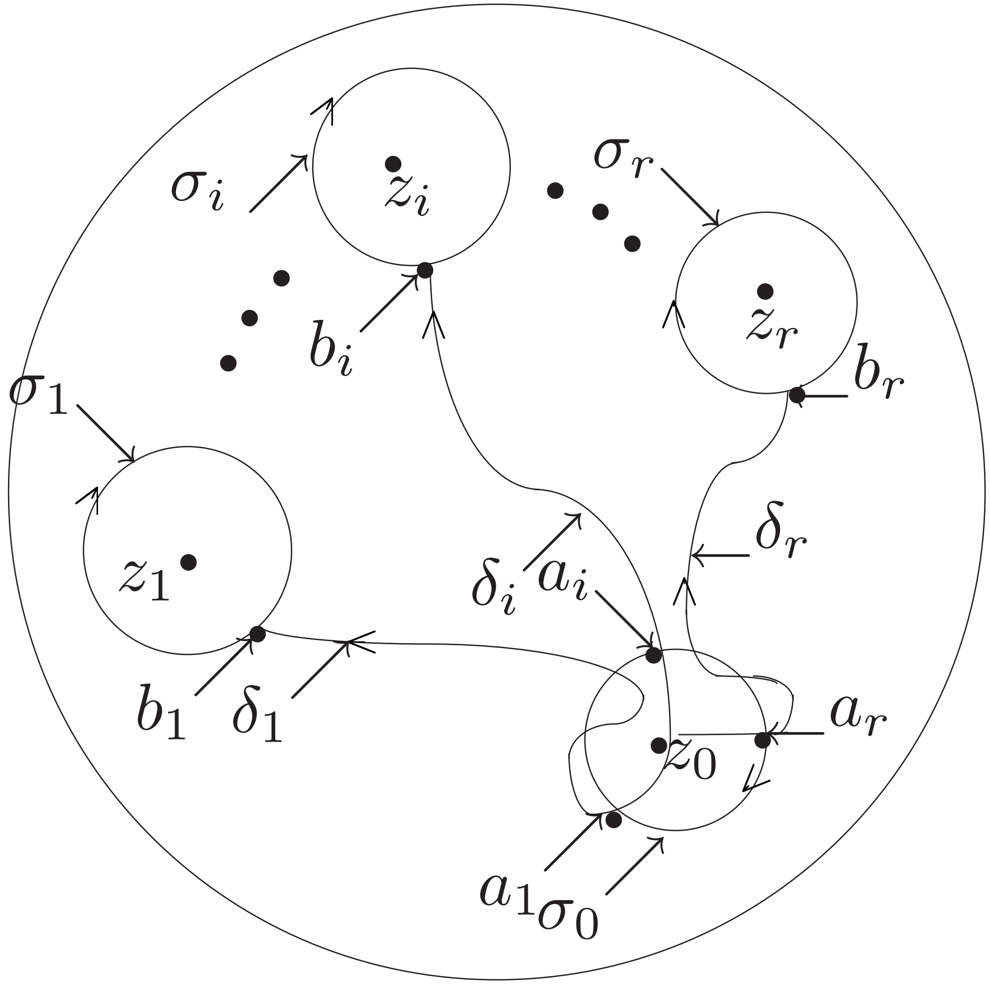
When $r = 4$, MT levels (start from level 0) are upper half-plane quotients covering classical j-line. Rarely are they modular curves.

With $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$, denote:
$\langle g_2, g_3 \rangle = H_{2,3}(\boldsymbol{g})$ and $\langle g_1, g_4 \rangle = H_{1,4}(\boldsymbol{g})$.

Modular curves have

- p-cusps: p divides $\mathbf{mpr}_{\boldsymbol{g}}$; and
- g-p' cusps: both $H_{2,3}$ and $H_{1,4}$ are p' groups.

Don't have: *o(nly)-p'* cusps neither p nor g-p'.



App. A_1: Classical Generators: see next 2 pages

Pieces in the figure

Ordered closed paths $\delta_i \sigma_i \delta_i^{-1} = \bar{\sigma}_i$, $i = 1, \ldots, r$, are *classical generators* of $\pi_1(U_{\boldsymbol{z}}, z_0)$.

Discs, $i = 1, \ldots, r$: D_i with center z_i; all disjoint, each excludes z_0; b_i be on the boundary of D_i.

Clockwise orientation: Boundary of D_i is a path σ_i with initial and end point b_i; δ_i a simple *simplicial* path: initial point z_0 and end point b_i. Assume δ_i meets none of $\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_r$, and it meets σ_i only at its endpoint.

Meeting Boundary of D_0

D_0 intersections: D_0 with center z_0; disjoint from each $D_1, \ldots, D_r$. Consider a_i, first intersection of δ_i and boundary σ_0 of D_0.

Crucial ordering: Conditions on $\delta_1, \ldots, \delta_r$:

- pairwise nonintersecting, except at z_0; and
- $a_1, \ldots, a_r$ are in order clockwise around σ_0.

Since paths are simplicial, last condition is independent of D_0, for D_0 sufficiently small.

App. B_1: R(iemann)-H(urwitz)

R-H: Computes the genus g_X of a degree n cover $\varphi : X \to \mathbb{P}^1_z$ from these ingredients.

- $\boldsymbol{z}_\varphi$ are the branch points, and $(\bar{\sigma}_1, \dots, \bar{\sigma}_r)$ are classical generators (App.A_1) of $\pi_1(U_{\boldsymbol{z}_\varphi})$.
- $X^0 = \varphi^{-1}(U_{\boldsymbol{z}_\varphi})$. So, $\varphi^0 : X^0 \to U_{\boldsymbol{z}}$ is unramified, giving $\varphi_* : \pi_1(U_{\boldsymbol{z}_\varphi}) \to S_n$.
- $\varphi_*(\bar{\sigma}_1, \dots, \bar{\sigma}_r) = (g_1, \dots, g_r)$.

Branch cycles and the genus

With $\mathrm{ind}(g_i) = n - |g_i \text{ orbits}|$,

$$2(n+g_X-1)=\sum_{i=1}^{r}\mathrm{ind}(g_i).$$

Then, $(g_1,\dots,g_r)$ are *branch cycles* of φ.

Exercise: Compute genus of a cover with branch cycles $\boldsymbol{g} \in \mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{abs}}$ in §I.B (p. 7). Same for $\boldsymbol{g} \in \mathrm{Ni}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{in}}$.

App. C_1: Apply R-H to MT components $(r=4)$

O' is a $\bar M_4$ orbit on a reduced Nielsen class $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}/\mathcal{Q}''$ (or $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}/\mathcal{Q}''$). Denote action of $(\gamma_0 = q_1q_2, \gamma_1 = \mathbf{sh}, \gamma_\infty = (\gamma_0\gamma_1)^{-1})$ on O' by $(\gamma_0', \gamma_1', \gamma_\infty')$: Branch cycles for a cover $\bar{\mathcal{H}}' \to \mathbb{P}^1_j$,

R-H gives genus, $g_{\bar{\mathcal{H}}'}$:

$$2(\deg(\bar{\mathcal{H}}'/\mathbb{P}^1_j)+g'-1)=\mathrm{ind}(\gamma_0') + \mathrm{ind}(\gamma_1') + \mathrm{ind}(\gamma_\infty').$$

App. D_1: Branch Cycle Argument for $(G, \mathbf{C})$

$f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ means f analytic around z_0 is *extensible* along all paths in $U_{\boldsymbol{z}}$ with limits in $\mathbb{C} \cup \{\infty\}$ as it approaches any $z' \in \boldsymbol{z}$. Let $z_0 \in \mathbb{Q}$.

Quest. A: Given $\boldsymbol{z}$, $\exists\, \varphi_{\boldsymbol{g}} : X_{\boldsymbol{g}} \to \mathbb{P}^1_z$ over $\mathbb{Q}$, $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ with branch points $\boldsymbol{z}$?

Quest. B: As above, but $\exists\, \varphi_{\boldsymbol{g}}$, Galois, over $\mathbb{Q}$?

Q. A or B requires that $\boldsymbol{z}$ is a $\mathbb{Q}$ set

So, $\sigma \in G_{\mathbb{Q}}$ acts on $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$ through what $\sigma^{-1} \circ \gamma \circ \sigma$ does to $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$:

$$f \mapsto f_{\sigma^{-1}\circ\gamma\circ\sigma} \stackrel{\mathrm{def}}{=} f_{\gamma^\sigma} \in \pi_1(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}.$$

Profinite $\pi_1(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ is free on r (topological) generators modulo a product-one relation.

Notation: $\sigma \in G_K$ maps to $n_\sigma \in \hat{\mathbb{Z}}^* = G(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$.

Branch Cycle Argument

For each $\sigma \in G_{\mathbb{Q}}$, let $\pi_\sigma \in S_r$ satisfy $z_i^\sigma = z_{(i)\pi}$.
Affirmative to Q. B: Requires

$$C_{(i)\pi_\sigma}^{n_\sigma} = C_i,\ \ i = 1, \ldots, r.$$

Affirmative for Q. A: Only requires some *Galois closure group* $G \le \hat{G} \le N_{S_n}(G, \mathbf{C})$: with

$$g_\sigma C_{(i)\pi}^{n_\sigma} g_\sigma^{-1} = C_i,\ \ i = 1, \ldots, r,\ \text{for some } g_\sigma \in \hat{G}.$$

Some Branch Cycle Argument Examples

Let $G = A_5$, C_5^+ the class of $(1\,2\,3\,4\,5)$, C_5^- the class of $(1\,3\,5\,2\,4)$, C_3 the class of 3-cycles.

"Yes" means for some $\boldsymbol{z}$:

1. $\mathbf{C}_{5_+^2 3^2}$: No for Q. B, yes for Q. A.
2. $\mathbf{C}_{5_+ 5_- 3^2}$: Yes for Q. A and B.
3. $\mathbf{C}_{5_+^2 5_-^2}$: Yes for Q. A and B.

App.E$_1$. **sh**-incidence matrix: $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{*,\mathrm{rd}}$, $*=\mathrm{abs}/\mathrm{in}$

A Nielsen class gives a space. Since $r = 4$, each component is — like modular curves — an upper half-plane quotient by a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z}) = H_4/\mathcal{Q}''$, ramified over $\{0, 1, \infty\} \subset \mathbb{P}^1_j$.

sh-incidence pairing on cusps: Given ${}_{\mathbf{c}}O_1, {}_{\mathbf{c}}O_2$ two (inn. or abs.) cusp orbits:
$({}_{\mathbf{c}}O_1, {}_{\mathbf{c}}O_2) \mapsto$ |distinct (inn. or abs.) Nielsen classes| from elements of ${}_{\mathbf{c}}O_1 \cap ({}_{\mathbf{c}}O_2)$**sh**.

Symmetric, since $\mathbf{sh}^2 \in \mathcal{Q}''$.

sh-incidence blocks $\leftrightarrow$ components of the space.

sh-incidence for $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$ ($k = 1$ on pg. 17)

Cusp orbit	${}_{\mathbf{c}}O_{p^2}$	${}_{\mathbf{c}}O_{a,p}, a \in (\mathbb{Z}/p)^*$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_{p^2}$	$p(p-1)$	1	1
${}_{\mathbf{c}}O_{a,p}, a \in (\mathbb{Z}/p)^*$	1	0	0
${}_{\mathbf{c}}O_1$	1	0	0

$u = 0 \leftrightarrow$ width p^2 H-M rep. cusp, ${}_{\mathbf{c}}O_{p^2}$;

$u = 1 \leftrightarrow$ cusps ${}_{\mathbf{c}}O_{a,p}$, $a \in (\mathbb{Z}/p)^*$ of width 1

(even though absolute p-cusps); and

$u = 2 \leftrightarrow$ width 1 cusp ${}_{\mathbf{c}}O_1$ of the shift of the H-M rep.

Adding to this data the fixed points of $\gamma_0 = q_1q_2$ and $\gamma_1 =$ **sh** gives the genus of the space (App. C_1).

Adjustments for $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{\mathrm{in,rd}}$ ($k=1$ on pg. 17)

$u=0 \leftrightarrow \varphi(p^2)/2 = \frac{p(p-1)}{2}$ $\mathrm{H\text{-}M}$ inner cusps over the unique absolute $\mathrm{H\text{-}M}$ cusp.
$u=2 \leftrightarrow$ Story the same for shifts of $\mathrm{H\text{-}M}$ cusps (width 1).
$u=1 \leftrightarrow$ over each such absolute (width 1) cusp are

$$\varphi(p)/2 = \tfrac{p-1}{2} \text{ width } p \text{ cusps.}$$

The sh-incidence matrix remains the same if we replace $\gamma_1 = \mathbf{sh}$ by γ_0 [Fr07b, Lem. 4.8]. So, fixed points of either are represented on the diagonal.

Problem: Use the MT method to compute which elements in $\mathrm{H\text{-}M}$ cusp are fixed points of γ_i, $i=0,1$.