Connectedness of spaces of Riemann Surface covers

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Connections between these topics

- R(egular)I(nverse)G(alois)P(roblem), and
- S(trong)T(orsion)C(onjecture) on abelian varieties

arise from inspecting properties of Hurwitz spaces: families of sphere covers of a specific type.

The I(nverse)G(alois)P(roblem): Is finite group G the Galois group of an extension of every number field?

Number Theory $\leftrightarrow$ Geometry $\leftrightarrow$ Combinatorics

The R(egular)IGP: Is there one Galois extension $L_G/\mathbb{Q}(z)$ with group G containing only $\mathbb{Q}$ for constants? From Hilbert's irreducibility Theorem, RIGP $\Longrightarrow$ IGP. The RIGP has provided most successes through *braid monodromy* ([Fr77], [FrV]).

Riemann's Existence Theorem: A combinatorial tool for describing regular extensions over $\bar{\mathbb{Q}}$. Describing Hurwitz spaces is a big RET.

Part I: Conjugacy classes and covers

G a group, **C** is r conjugacy classes in G.

- $\boldsymbol{g} = (g_1, \dots, g_r) \in$ **C** means $g_{(i)\pi}$ is in C_i, for some π permuting $\{1, \dots, r\}$.
- $\Pi(\boldsymbol{g}) \stackrel{\text{def}}{=} \prod_{i=1}^{r} g_i$ (order matters).

An analytic cover, $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces, ramifies over a finite set of points $\boldsymbol{z} = z_1, \dots, z_r \subset \mathbb{P}^1_z$: $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}$.

Then, $\varphi \implies (G, \mathbf{C}, \boldsymbol{z}), G \le S_n$, with $n = \deg(\varphi)$: G the *monodromy group of* φ.

Nielsen classes/ R(iemann's)E(xistence)T(heorem)

Fix $\boldsymbol{z} = \boldsymbol{z}^0$ and *classical generators* of $\pi_1(U_{\boldsymbol{z}^0}, z_0)$.

Combinatorial description of all $\varphi \implies (G, \mathbf{C})$:

Nielsen classes:

$$\{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \Pi(\boldsymbol{g}) = 1\} \stackrel{\text{def}}{=} \mathrm{Ni}(G, \mathbf{C}).$$

Projective r space $\mathbb{P}^r \Leftrightarrow$ degree $\le r$, monic polynomials; deg $< r-1$ or with equal zeros form its *discriminant* locus D_r. Denote $\mathbb{P}^r \setminus D_r$ by U_r.

Hurwitz combinatorics: Deformations (r branch points) of $\varphi \implies$ paths in U_r based at $\boldsymbol{z}^0$.

One cover defines a family: $\varphi : X \to \mathbb{P}^1_z \implies$

1. Permutation representation of $\pi_1(U_r, \boldsymbol{z}^0) \stackrel{\text{def}}{=} H_r$ *Hurwitz monodromy* on orbit Ni'_φ —independent of classical generators —of $[\varphi] \in \text{Ni}(G, \mathbf{C})$.

2. An unramified connected cover $\mathcal{H}(G, \mathbf{C})_\varphi \to U_r$: Hurwitz space component containing φ.

Equivalences of covers and Nielsen classes.

[Abs.]$\varphi' : X' \to \mathbb{P}^1_z \sim \varphi \Leftrightarrow \boldsymbol{g} = h\boldsymbol{g}'h^{-1}, h \in N_{S_n,\mathbf{C}}(G)$.

[Inn.]φ Galois with $\mu : \text{Aut}(X/\mathbb{P}^1_z) \xrightarrow{\text{isom}} G \sim (\varphi', u') \Leftrightarrow$ $\boldsymbol{g} = h\boldsymbol{g}'h^{-1}, h \in G$.

Part II: Two Connectedness Results:

II.A. Constellations of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ [AGLI, §1]

$\xrightarrow{g\geq 1}$	$\ominus\ \oplus$	$\ominus\ \oplus$	...	$\ominus\ \oplus$	$\ominus\ \oplus$	$\xleftarrow{1\leq g}$
$\xrightarrow{g=0}$	$\ominus$	$\oplus$	...	$\ominus$	$\oplus$	$\xleftarrow{0=g}$
$n \geq 4$	$n = 4$	$n = 5$	...	n even	n odd	$4 \leq n$

Theorem 1 (tag $\xrightarrow{g=0}$, $r = n-1$, $n \geq 5$). $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\rm in}$ *has one component. Further,* $\Psi_{\rm abs}^{\rm in} : \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\rm in} \to \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\rm abs}$ is deg. 2.

Theorem 2 (tag $\xrightarrow{g\geq 1}$, $r \geq n \geq 5$). $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\rm in}$ *has two components,* $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\rm in}$ *(symbol $\oplus$) and* $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\rm in}$ *(symbol $\ominus$). Further*

$$\Psi_{\rm abs}^{\rm in,\pm} : \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\rm in} \to \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\rm abs} \text{ has degree 2.}$$

For $n = 4$, two 3-cycle classes C_{+3}, C_{-3} in A_4, $\mathbf{C} = \mathbf{C}_{+3^{s_1}\cdot -3^{s_2}}$: $\mathrm{Ni}(G, \mathrm{C}_{\pm 3^{s_1,s_2}})$ nonempty iff

$$s_1 - s_2 \equiv 0 \bmod 3 \text{ and } s_1 + s_2 = r.$$

Frattini covers

Frattini cover $G' \to G$ is a group cover (surjection) with restriction to a proper subgroup not a cover. Get a lifting invariant from a *central* Frattini cover.

Central Frattini from A_n: Spin_n^+ the nonsplit degree 2 cover of the connected component O_n^+ of the orthogonal group. Regard $S_n \subset O_n$; $A_n \subset O_n^+$. Denote pullback of A_n to Spin_n^+ by Spin_n. Identify $\ker(\mathrm{Spin}_n \to A_n)$ with $\{\pm 1\}$.

F-S Small lifting invariants ([LUM,§1], [Ser90a])

Odd order $g \in A_n$ has a unique odd order lift, $\hat{g} \in \mathrm{Spin}_n$. Let $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C})$ with $\mathbf{C}$ odd-order. *Small* lifting invariant:

$$s(\boldsymbol{g}) = s_{\mathrm{Spin}_n}(\boldsymbol{g}) = \hat{g}_1 \cdots \hat{g}_r \in \{\pm 1\}.$$

For g odd-order, let $w(g)$ by the number of cycles in g with lengths (ℓ) with $\frac{\ell^2-1}{8} \equiv 1 \mod 2$.

Theorem 3 (F-S). *On any braid orbit, $s(\boldsymbol{g})$ is constant (explains Const. diag. comps). If genus 0 Nielsen class, then $s(\boldsymbol{g}) = (-1)^{\sum_{i=1}^r w(g_i)}$.*

II.B. Pure-cycle components

- $g \in S_n$ is *pure-cycle* if one cycle has length > 1.
- Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ is *pure-cycle* if all conjugacy classes are pure-cycle (a d-cycle).
- If $d_1, \dots, d_r$ are the pure-cycle lengths, denote the Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_r})^*$ (* an equivalence).

Assume $G \le S_n$ transitive and $\mathbf{C}^{S_n} \stackrel{\mathrm{def}}{=} \mathbf{C}_{d_1 \cdots d_r}$ image of $\mathbf{C}$ in S_n, with d_i s all odd. Necessary condition $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ is nonempty: Genus

$$\mathbf{g} = \mathbf{g}_{d_1 \cdots d_r} \stackrel{\mathrm{def}}{=} \frac{\sum_{i=1}^{r} d_i}{2} - (n-1) \text{ is non-negative.}$$

Liu-Osserman genus 0 result [LOs06]

Theorem 4. *If* $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}_{d_1\cdots d_r})$ *has genus 0, then* $G = A_n$, *and* H_r *is transitive on it.*

Compactify the reduced inner space:

$$\bar{\mathcal{H}}(A_n, \mathbf{C}_{d_1\cdot d_2\cdot d_3\cdot d_4})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \bar{\mathcal{H}}_{n,d_1\cdots d_4}.$$

Consider $\{\bar{\mathcal{H}}(G_k(A_n), \mathbf{C}_{d_1\cdot d_2\cdot d_3\cdot d_4})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \bar{\mathcal{H}}_{n,d_1\cdots d_4,k}\}_{k=0}^{\infty}$ with $G_k(A_n) \to A_n$ the universal exponent 2^k 2-group extension of A_n.

Goal ($r = 4$): Decide if genuses of components grow with k. Assume: Exists a g-$2'$ cusp $\implies$ all d_i s same ($= d$). Genus 0 $\implies$ $2(d-1) = n-1$.

Inner (resp. absolute) Reduced spaces [BFr02, §2]

Reduced equiv.: $\varphi : X \to \mathbb{P}^1_z \sim \beta \circ \varphi, \beta \in \mathrm{PGL}_2(\mathbb{C})$.

j-invariant: $\boldsymbol{z} \in U_4 \mapsto j_{\boldsymbol{z}} \in U_\infty \stackrel{\mathrm{def}}{=} \mathbb{P}^1_j \setminus \{\infty\}$ of $\boldsymbol{z}$. Normalize so $j = 0$ and 1 are *elliptic points*: $j_{\boldsymbol{z}}$ with more than a Klein 4-group stabilizer in $\mathrm{PGL}_2(\mathbb{C})$.

Reduced classes of covers with j-invariant $j' \in U_\infty$ $\Leftrightarrow$ elements of reduced Nielsen classes.

Part III: $r = 4$ Upper-half plane quotients

Recall: $H_4 = \langle q_1, q_2, q_3 \rangle$: Acts on any Nielsen classes with $r = 4$ by a twisting on its 4-tuples:

$$q_2 : \boldsymbol{g} \mapsto (\boldsymbol{g})q_2 = (g_1, g_2 g_3 g_2^{-1}, g_2, g_4).$$

Reduced equivalence corresponds to modding out the Nielsen class by $\mathcal{Q}'' = \langle (q_1 q_2 q_3)^2, q_1 q_3^{-1} \rangle \le H_4$.

H_4 on reduced Nielsen classes factors through the *mapping class group*: $\bar{M}_4 \stackrel{\text{def}}{=} H_4/\mathcal{Q}'' \equiv \mathrm{PSL}_2(\mathbb{Z})$.

III.A. Using generators of $\bar{M}_4$

$\bar{M}_4 = \langle \gamma_0, \gamma_1, \gamma_\infty \rangle, \gamma_0 = q_1 q_2$ (order 3),
$\gamma_1 =$ **shift** $= q_1 q_2 q_3$ (order 2),
$\gamma_\infty = q_2$ ($j = \infty$ monodromy generator),
satisfying the product-one relation: $\gamma_0 \gamma_1 \gamma_\infty = 1$.
The *cusp group* $\mathrm{Cu}_4 = \langle q_2, \mathcal{Q}'' \rangle \le H_4$:

A cusp is an orbit of Cu_4. $(\boldsymbol{g})$**sh** $\mapsto$ reduced class of (g_2, g_3, g_4, g_1). and **sh**^{2} is trivial.

Riemann-Hurwitz on components

Interpret R-H: Denote $(\gamma_0, \gamma_1, \gamma_\infty)$ acting on Ni_{d^4} as giving branch cycles for $\bar{\mathcal{H}}_{d^4} \to \mathbb{P}^1_j$. Denote the resulting permutations by $(\gamma_0', \gamma_1', \gamma_\infty')$:

- Points over 0 (resp. 1) $\Leftrightarrow$ orbits of γ_0 (resp. γ_1).
- The index contribution $\mathrm{ind}(\gamma_\infty)$ from a cusp with rep. $\boldsymbol{g} \in \mathrm{Ni}_{d^4}$ is $|(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}''| - 1$.

2-Frattini extensions of A_5

$(\mathbb{Z}/2)^2 \times^s \mathbb{Z}/3 = A_4$: The universal 2-Frattini extension of A_4 is ${}_2\tilde{G}(A_4) = \tilde{F}_2 \times^s \mathbb{Z}/3$.

Univ. 2-Frattini extension ${}_2\tilde{G}(A_5)$ of A_5: Restriction over A_4 is ${}_2\tilde{G}(A_4)$. With $\ker_0 = \ker({}_2\tilde{G}(A_5) \to A_5)$,
$\Phi_1(\ker_0) = \langle (\ker_0, \ker_0), \ker_0^2 \rangle$.
Then, $\Phi_k(\ker_0) \stackrel{\text{def}}{=} \Phi_{k-1}(\Phi_1(\ker_0))$.

Iterate Φ_1 to get max. exp. 2^k Frattini extension of A_5: $G_k(A_5) \stackrel{\text{def}}{=} {}_2\tilde{G}(A_5)/\Phi_k(\ker_0)$.

III.B. Modular curve-like towers $\{\bar{\mathcal{H}}(G_k(A_5), \mathbf{C}_{3^4})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$

Ram_{r_0}: Choose any r_0. For $k \geq 0$, use covers in $\mathrm{Ni}(G_k, \mathbf{C}_k)$ with at most r_0 classes in $\mathbf{C}_k$.

Question 5 (RIGP(A_5,p=2,r_0) Quest.). Is there r_0, so the RIGP holds for all G_k s from covers in Ram_{r_0}?

Theorem 6. *If the answer is "Yes!," then there are $2'$ conjugacy classes* $\mathbf{C}$ *(no more than r_0) in G, and a projective system $\{\mathcal{H}'_k \subset \mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$ (a Modular Tower component branch over $\mathbb{Q}$) each having a $\mathbb{Q}$ point ([D06] [FrK97]).*

The Main Conjecture

Conjecture 7 (MainConj.). If $k >> 0$, $\mathcal{H}'^{\rm rd}_k(\mathbb{Q}) = \emptyset$.

Our examples: Towers over $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$, odd $n \ge 5$, $p = 2$. Three cusp types [LUM, §3]:

$$H_{2,3}(\boldsymbol{g}) \stackrel{\rm def}{=} \langle g_2, g_3\rangle \text{ and } H_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle;$$

and $(\boldsymbol{g})$**mpr** $\stackrel{\rm def}{=} \mathrm{ord}(g_2g_3)$, *middle product order*.

- *p cusps*: $p\big|(\boldsymbol{g})$**mpr**.
- *g(roup)-p'*: $H_{2,3}(\boldsymbol{g})$ and $H_{1,4}(\boldsymbol{g})$ are p' groups.
  H-M rep.: $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1}) \implies (\boldsymbol{g})$**sh** is g-$p'$.
- *o(nly)-p'*: $p \not| (\boldsymbol{g})$**mpr**, but the cusp is not g-p'.

III.C. **sh**-incidence for $r = 4$ and $\mathrm{Ni}_{(\frac{n+1}{2})^4}$

$(\boldsymbol{g})$**mpr**: (g_2, g_3) pairs for abs. cusp reps.:

n: H-M rep.: $(\bullet, (1 \ \dots \ \frac{n+1}{2}), (\frac{n+1}{2} \ \frac{n+3}{2} \ \dots \ n), \bullet)$

$n{-}2$: $(\bullet, (2 \ \dots \ \frac{n-1}{2} \ \frac{n+3}{2} \ \frac{n+1}{2}), (\frac{n+1}{2} \ \frac{n+3}{2} \ \dots \ n), \bullet)$

$\dots$

1: shift of H-M rep.: $(\bullet, (\frac{n+1}{2} \ \frac{n+3}{2} \ \dots \ n)^{-1}, (\frac{n+1}{2} \ \frac{n+3}{2} \ \dots \ n), \bullet$

1. Fill in $\bullet$s (1st and last rows hint how), and apply Cu_4.
2. q_2 orbit length is $2 \cdot (\boldsymbol{g})$**mpr** unless $(\boldsymbol{g})$**mpr** $= o$ odd, and $\mathrm{ord}((g_2g_3)^{\frac{o-1}{2}} g_2) = 2$ [BFr02, Prop. 2.17]. Latter holds:
$$\deg(\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}/\mathbb{P}^1_j) = (\frac{n+1}{2})^2.$$
3. All level 0 L-O cusps are H-M or o-$2'$.

sh-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{34}^{\mathrm{in,rd}}$

Pairing on Cu_4 orbits: $(O, O') \mapsto |O \cap (O')\mathbf{sh}|$. $O_{5,5;2}$ (resp. $O_{1,2}$) indicates 2nd **mpr** 5, width 5 (resp. only **mpr** 1, width 2) orbit. **sh**-incidence gives $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ genus.

Orbit	$O_{5,5;1}$	$O_{5,5;2}$	$O_{3,3;1}$	$O_{3,3;2}$	$O_{1,2}$
$O_{5,5;1}$	0	2	1	1	1
$O_{5,5;2}$	2	0	1	1	1
$O_{3,3;1}$	1	1	0	1	0
$O_{3,3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

Complete orbit for $\bar{M}_4 = \langle \mathbf{sh}, \gamma_\infty \rangle$ on $\mathrm{Ni}_{34}^{\mathrm{in,rd}}$ in 2-steps: Apply $(\mathbf{sh} \circ \mathrm{Cu}_4)^2$ to H-M rep.

Frattini Principles [LUM, §3]

A MT is defined by a projective sequence $\{\mathrm{Ni}'_k\}_{k=0}^{\infty}$ of H_r orbits on $\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in,rd}} \implies$ there is a projective sequence of cusp reps (cusp branch).

[FP1] A p cusp at level k_0 has above it at level k only p cusps of width increased by p^{k-k_0}.

[FP2] g-$2'$ cusp at level 0 $\implies$ g-$2'$ cusp branch.

[FP3] Lifting invariant gives iff test for all cusps above level k o-p' cusps being p cusps ([LUM, §4], [We]).

Cusp Tree Conclusions in Liu-Osserman cases

[STMT] Strong Tors. Conj. $\Longrightarrow$ Main MT Conj. and ($\sim\Leftrightarrow$).

Apply F-S lift inv. to $(g_2, g_3, (g_2g_3)^{-1})$ for Ni_{34}: Level 0 o-$2'$ cusps $O_{5,5,\bullet}$ and $O_{3,3,\bullet}$ have only 2 cusps above them: $(A_5, \mathbf{C}_{3^4}, p = 2)$ cusp tree has only g-$2'$ or 2 cusp branches.

Theorem 8. *If* ≥ 3 p *cusps for* MT *level* $k >> 0$ $\Longrightarrow$ *Main Conj* $\Longrightarrow$ *holds for L-O cases (many 2 cusps at level 1). If a cusp branch is both H-M and* p*, then* MT *cusp tree contains a spire: a modular curve cusp tree. At level 1, holds for (L-O)* $n = 5$*, but not for* $n = 9$*.*

Question 9. When does it hold for Fried + L-O cases?

Abbreviated References: [LUM] has much more

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[Def-Lst]Select from the list in www.math.uci.edu/conffiles_rims/deflist-mt/full-deflist-mt.html of present MT-related definitions. 09/05/06 examples: Branch-Cycle-Lem CFPV-Thm Cusp-Comp-Tree FS-Lift-Inv Hurwitz-Spaces Main-MT-Conj Modular-Towers Nielsen-Classes RIGP Strong-Tors-Conj mt-rigp-stc p-Poincare-Dual sh-Inc-Mat. A similar URL, www.math.uci.edu/conffiles_rims/deflist-mt/full-paplist-mt.html, is a repository for not just mine, but also of the growing list of those joining the MT project.

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