Combinatorics of Sphere Covers and the Shift-Incidence Matrix

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1. The mathematical objects of the proposal

Most problems in algebra, at some stage, must consider algebraic equations. Sometimes solving them, other times producing them – so their existence manifests a desired structure.

Akin to doing differential equations, a researcher in algebraic equations must specialize to get somewhere. My speciality is algebraic equations in two variables, say w and z, where one variable, say z, interprets as a data variable. In this proposal I take relations with coefficients in the complex numbers $\mathbf{C}$.

Such an algebraic equation belongs to a unique equivalence class of compact covers of the Riemann sphere, $\mathbb{P}^1_z = \mathbf{C}_z \cup \{\infty\}$ (with $\mathbf{C}_z$ the standard z-plane of complex variables). The word cover here means a (ramified) analytic (nonconstant) map from a compact Riemann surface to the sphere. The notation for such a cover is $\varphi : X \to \mathbb{P}^1_z$. Points $\{z_1, \dots, z_r\}$ over which the cover ramifies are called *branch points*.

1.1. Overall proposal goals.

1.1. Overall proposal goals. Two such covers $\varphi_i : X_i \to \mathbb{P}^1_z$, $i = 1, 2$, are *absolute* equivalent if there is a continuous $\psi : X_1 \to X_2$ that commutes with the φ_i s. Suppose these are Galois covers with group $G(X_i/\mathbb{P}^1_z)$, having a particular isomorphism α_i with a finite group G, $i = 1, 2$. Then, ψ induces a homomorphism $\psi^* : G(X_1/\mathbb{P}^1_z) \to G(X_2/\mathbb{P}^1_z)$. We call ψ an *inner* equivalence of the (φ_i, α_i) s if in addition $\alpha_2 \circ \psi^* \circ \alpha_1^{-1}$ is conjugation by an element of G.

This proposal uses the language of covers, their absolute and inner equivalence classes, and – in each case – *reduced* equivalence, where you add an action of Möbius transformations on $\mathbb{P}^1_z$. In this section I label some general strategic goals as $\mathbf{Goal}_n$, $n = 1, \dots, 4$. Example: $\mathbf{Goal}_1$ appears right after the two main tools (1.2) for this proposal. Later sections formally list tactical goals corresponding to technical elements in the strategic goals.

My sub-speciality is producing equations to solve existence questions. Work with Pierre Debes showed how to interpret many papers using the algebraic equations called *modular curves* (or their hyperelliptic analogs) as special cases of the Inverse Galois problem: the part involving so-called *involution realizations* of dihedral groups [**DFr94**, §5.2].

I recognized that the intense study of modular curves had led to many beautiful tools. So, I found a way to see the whole Inverse Galois Problem as a generalization of their study [**Fr95**] by making precise conjectures on the objects – M(odular) T(owers) – that fulfill this generalization.

Each MT is constructed from a finite group G, some conjugacy classes of G and a prime p. A *Nielsen class* §1.4 is the precise data for a Hurwitz space, and to define the MTs over such a space requires an extra condition. The MT then consists of a projective system of components of reduced Hurwitz spaces (§1.4), as do towers of modular curves (though not the standard way to see them). The idea was to extend conjectures appreciated by modular curve people to where we could replace dihedral groups by essentially all p-perfect finite groups, p any prime (§1.3).

Recently, these conjectures have seen three separate chunks of progress using the geometry of Modular Tower levels.

(1.1a) The Main MT Conjecture [**FrK97**] is that high tower levels have no points over a fixed number field. Cadoret-Tamagawa showed this for any MT defined by $r \le 4$ conjugacy classes [**CadT08**].
(1.1b) Let G be any p-perfect finite group. Based on Harbater patching [**Ha84**] and [**Fr95**, Thm. 3.21], Debes-Emsalem produce a MT over $\mathbb{Q}$ attached to $(G, \mathbf{C}, p)$ ($r \ge 5$, varies with G) having a projective system of $\mathbb{Q}_\ell$ points for all primes ℓ not dividing $|G|$ [**DEm04**, Rem. 4.2.b].
(1.1c) For all $n \equiv 5 \mod 8$, [**Fr08a**, Cor. 5.17] produces MTs for $G = A_n$, $p = 2$ and $r = 4$, whose cusp tree contains a subtree – a *spire* (Princ. 5.6) – isomorphic to the cusp tree for modular curve towers.

Results (1.1a) and (1.1c) refer to $r = 4$ branch point covers. Then, the reduced Hurwitz space component at each MT level is an upper half-plane quotient by a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ [**BFr02**, Prop. 3.28]. Only for modular curve towers is this a congruence subgroup.

My approach has been to show high tower levels have *general type* – a statement about sections of the sheaf of holomorphic differentials on the compactification of the tower level. I do this by proving cusp properties at the lowest tower level (level 0). An invariant, the **sh**-incidence matrix (1.2a), includes a display of elliptic ramification and cusps. The §4.2 examples, applied to (1.1c), graphically show the genus of these level 0 curves.

This mimics how one might study a specific modular curve, except I replace the dihedral groups that underlay modular curves by alternating groups A_n, $n \equiv 1 \mod 4$.

Following [**Fr06**, Thm. 5.1], we seek 2-cusps. These §4.2 examples have none at level 0. Yet, Princ. 5.6, still using level 0 cusps, shows 2-cusps appear at level 1, and their numbers grow with the levels. So, [**Fr08b**] is explicit (over-kill really) for the Main Conjecture in the cases it covers.

The [**CadT08**] approach is very different – not at all explicit about properties of levels, and totally number theoretic. While it does the Main Conjecture for all cases when $r = 4$, unlike my approach it offers no generalization to the full problem.

My approach is based on connectedness results. §3.1 gives five examples, two over 140 years old, two very recent. §3.2 reminds of the [**FV91**] reason for such results: Finding components with definition field $\mathbb{Q}$. This proposal's connectedness goal is subtler.

It is to assure a component boundary has two precise types of cusps, each identifying a distinct component property. Some applications generalize Serre's use of the *width* (ramification index, head of §4) 1 and p cusps on $X_0(p)$ in [**Ser68**]. Two tools show what I mean by being precise about using cusps (examples of both are in §4):

(1.2a) The sh(ift)-incidence matrix: a *cusp pairing* on all MT levels, including modular curves (but not on all upper half-plane quotients).
(1.2b) Adaptation to cusps of a Fried-Serre formula: This gives the "nature" of the cusps (a la §2.1).

Goal$_1$ is to extend the precise results of (1.1c) beyond the prime $p = 2$. *Each* alternating group example here produces examples involving all primes – sketched in §5.2.2. So, it gives a framework of towers as rich as does the whole collection of modular curve towers, including "Hecke-like operators."

[**Fr06**, §6] outlines why the spire in (1.1c) supports a version of *Serre's Open Image Theorem* (for modular curves) as in [**Ser68**]. That is an example outcome we can expect from fulfilling this proposal. Should this proposal be supported, **Goal**$_2$ is to publish that, now that results give confidence in the basic conjectures.

Fried-Voelklein [**FV91**] showed the precise correspondence between regular realizations and rational points on Hurwitz spaces. My web site recounts that it did, and still does, give many positive results. (1.1b) is just one example. In one fell swoop, each MT alluded to there contains points giving all p-Frattini extensions of G (§1.3) as regular realizations (yes, over almost all ℓ-adic fields and the reals). Yet, excluding finitely many G, *none* of those groups have been realized over $\mathbb{Q}$, except possibly by the braid monodromy method (§1.2), at level 0.

What (1.1b) exploits is having a precise MT over $\mathbb{Q}$: all its levels have definition field $\mathbb{Q}$. This exposes three possibilities for any Hurwitz space with an allowed Nielsen class (§1.4).

(1.3a) It is level 0 of some MT (Def. 1.1) over some number field.
(1.3b) It is level 0 of some MT, but the degree over $\mathbb{Q}$ of component definition fields rises with the level.
(1.3c) There is no MT over it (high levels must be nonempty).

Given a Nielsen class, the §2.2.3 lifting invariant ([**Fr95**, §III.D]; 1st case [**Ser90a**]) theoretically can decide if (1.3c) holds. [**Fr08b**, Prop. 3.10] does this for all the (allowed) Nielsen classes appearing in [**LOs08**]. Example: At Serre's prompting, [**Fr95**, Ex. 3.13] showed the Nielsen classes for $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})$, and $p = 2$, support no MT for any even $n \geq 6$, but do for all odd n.

As §2.2.3 recounts, the (1.2) tools also help check for existence of p-cusps. This is the technical ingredient that makes [**Fr08b**] work. Though problems left in (1.3c) are nontrivial, they are technical. They aren't this proposal's main concern because our present goals can exploit examples where those technical details disappear.

That leaves the distinction between (1.3a) and (1.3b). That must be a hard problem. Even Serre and Shimura ran into analog difficulties in their situations: it gets entangled with interpretations of complex multiplication. **Goal**$_3$ is to figure the relation between our cusps types and that distinction. We suspect that conclusion (1.3a) follows from having *sufficiently many* p-cusps, likely akin to (1.1c) with its spire (so, the number of p-cusps grows with the level).

The difference between the results with $r = 4$ and $r > 4$ shows here. [**Fr95**, Thm. 3.21] has been the archetype result to produce MTs over $\mathbb{Q}$. This has been the only way, so far, to produce (1.3a) examples – outside modular curves. That criterion, however, never works for $r = 4$.

[**Cau08**] has relaxed the [**Fr95**, Thm. 3.21] hypotheses, to sometimes improve the lowest possible values of r for a particular group G that gives examples of (1.3a). Still, it will take a new idea to get $r = 4$, even for MTs (Princ. 2.4 produces them) over the Nielsen classes whose **sh**-incidence matrices §4.2 discusses. Yet, I don't know if they are examples of (1.3b). Resolving that is **Goal**$_4$.

1.2. Yes, the Inverse Galois Problem is of interest, but . . . The Inverse Galois Problem (IGP) for a finite group G asks if G is the Galois group of an extension of every number field. The R(egular)IGP improves this by seeking one Galois extension $L_G/\mathbb{Q}(z)$ with group G having just $\mathbb{Q}$ for constants. From Hilbert's irreducibility Theorem, RIGP $\implies$ IGP. The RIGP has provided most modern successes through *braid monodromy* from [**Fr77**], intensively applied in, say, [**MM99**], [**Vö96**] and [**Vö98**]. §3.2 explains more of the successes of the method.

By contrast, the Main MT Conjecture shows why such rational points won't exist where many thought they would. You can't produce such realizations even by magic or luck, if they don't exist. Wild guesses about what conjugacy classes to use have never worked.

It is really generalizations of the RIGP that have so many applications. Many problems require producing covers defined over one field whose Galois closures are defined over (possibly) much larger fields. (For example, the heart of the presentations of $G_{\mathbb{Q}}$ in [**FV92**].) My (recently revamped) website has html conversations on the success of the *monodromy method* in solving specific problems such as those fostered by Davenport's [**Fr08c**]. I've worked hard to clarify misunderstandings on the RIGP and to make it accessible. Many others have used it.

Indeed, I've written shell programs so that when asked about particular topics or definitions (such as how to use the Branch Cycle Lemma, see [**Fr08d**]), I return something useful semi-automatically, even if the requestor reads only the e-mail response. Here is a partial e-mail response directing someone to *monodromy method* material, illustrated by how it solved Davenport's Problem [**Fr08c**]. The full response would follow this pointer to direct the requestor to a particular section responding to their specific question.

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 On my home page http://www.math.uci.edu/~mfried/ at section: Ia. Home site
for Articles and Talks on these topics:
        http://www.math.uci.edu/~mfried/sectIa.html
 --> * Articles: Arithmetic of Covers (outside Modular Towers); I(nverse)G(alois)P(roblem):
Hurwitz spaces - their cusps and components, CFPV-connectedness, Hilbert-Siegel problems
http://www.math.uci.edu/~mfried/paplist-cov.html (has a comment button for each item)
 --> Item #24
Variables Separated Equations and Finite Simple Groups: (unabridged)
a complete version of UMStoryShort.html. The initiation of the monodromy method included
two new tools: theB(ranch)C(ycle)L(emma) and Hurwitz monodromy action. By walking through
Davenport's problem with hindsight, variables separated equations simplify lessons on
these tools. We use that to attend to these general questions:
(*) What allows us to produce branch cycles?
(*) What is in the kernel of the Chow motive map?
(*) What groups arise in 'nature?'
Each phrase addresses an aspect of formulating problems based on equations. That is,
we seem to need algebraic equations. Yet why, and what do we lose in using more easily
manipulated surrogates for them?

For your convenience the attached html and pdf file URLS goes directly to them:
http://www.math.uci.edu/~mfried/paplist-cov/UMStory.html
http://www.math.uci.edu/~mfried/paplist-cov/UMStory.pdf
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The Main Conjecture of MTs generalizes Mazur-Merel in the sense that all p-perfect finite groups generalize the dihedral group of order $2p$. That finishes my discussion relating the RIGP to Modular Towers. [**Fr07**] is an exposition to an audience trained in the shadow of Mazur-Merel.

The rest of this proposal discusses the group theory, combinatorics and geometry of the spaces featured in its title. There will be few further references to arithmetic geometry.

The goal is serious deformation invariants of sphere covers. The approach is to divine what are the components of maximal deformation of covers with r branch points, and what is the nature of the "cusps" that lie on the boundary of these components. So, one parameter leads them all, the value r. The next is its Nielsen class (§1.4).

The problem of components then comes to distinguishing the components within a given Nielsen class. §3 lists some connectedness results. This gives a context for the relating [**LOs08**] on pure-cycle genus 0 covers and [**Fr08a**] on 3-cycle covers.

The biggest idea in this proposal is how to handle cusps. Even in traditional cases, where congruence subgroups dominate, cusps handled classically are hard business. Instead, I found a finite group approach that effectively yields to combinatorial techniques.

To simplify I've described cusp types here just for $r = 4$. §2.1 has the definition of cusps. Especially of p-cusps and g-p' cusps, the kind that generalize those appearing on modular curves.

Liu and Osserman are combinatorialists who consulted with me. We were mutually helpful in that interchange. [**LOs08**] showed the *absolute* Hurwitz spaces of *pure-cycle genus 0 covers* have one connected component (see (3.1c)). The main result of [**Fr08a**] and their result [**LOs08**] have a conspicuous overlap: $r = n-1$ and 3-cycle pure-cycles. The case $n = 5$ is level 0 of the main MT that illustrates the [**BFr02**] theory.

I originally intended [**Fr08b**] to analyze the 2-cusps on all *odd pure-cycle* Liu-Osserman examples. Yet, I saw that what looked like uniform examples, included very different phenomena. So, [**Fr08b**, Main Thm.] goes deeper into two infinite MT lists whose level 0 are [**LOs08**] cases.

For example, §4.2 graphically illustrates the cusps and genuses of the *inner* Hurwitz spaces in the two lists: For one list there are two level 0 components, and for the other just one. Further, the nature of the 2-cusps in the MTs over them are very different: For one list there is a spire starting at level 1 (Princ. 5.6), and for the other, not.

Another example, level 1 (not level 0) of a modular curve ($X_0(p^2)$) tower, shows the contrast between cusps on an alternating group tower with those on a dihedral group tower.

1.3. Enough group theory to explain what is a Modular Tower. A profinite group is p-perfect if the prime p divides its order, but it has no $\mathbb{Z}/p$ quotient. Examples: Simple — not cyclic — groups of order divisible by p; dihedral groups $D_{p^{k+1}}$ of order $2p^{k+1}$, $k \ge 0$ and p odd; but not p groups, nor S_n for $p = 2$. Call a set p' if its elements have order prime to p. For example, a p' conjugacy class is all conjugates of a p' element.

Refer to a covering homomorphism $H \to G$ with no proper subgroup of H mapping onto G as a *Frattini cover*. Call it *p-Frattini* if the kernel is a p-group. Each p-perfect finite group G has infinitely many distinct p-Frattini extensions, all quotients of the universal p-Frattini cover ${}_p\tilde G$ of G. Denote p-adic integers by $\mathbb{Z}_p$.

The kernel, $\ker_0 \stackrel{\text{def}}{=} \ker_{G,p} = \ker({}_p\tilde G \to G)$, of ${}_p\tilde G \to G$ is a (nontrivial) finitely generated pro-free pro-p group [**FrJ04**, Chap. 22.11]. The abelianization of that kernel, $\ker_{G,p}/(\ker_{G,p}, \ker_{G,p})$ is then a $\mathbb{Z}_p[G]$ module $K_{\text{ab},p}$. I now say some words on its rank.

Frattini quotients whose map to G have kernel of exponent p pack together to give the universal exponent p cover $G_1 \to G$ (we suppress p). The kernel $V_{0,p} \stackrel{\text{def}}{=} \ker(G_1 \to G)$ is a $\mathbb{Z}/p[G]$ module. By the universality, $K_{\text{ab},p}/pK_{\text{ab},p} = V_{0,p}$. Denote the p profree group of rank t' by ${}_p\tilde F_{t'}$.

There is an "easy" (but see below) p-split case, where $G = P \times^s H$ with P a (normal) p-Sylow of rank (minimal number of generators) t'. Then, ${}_p\tilde G = {}_p\tilde F_{t'} \times^s H$.

If the center of G is prime to p, then G_1 has exactly the same center. This is a homological observation, equivalent to saying $V_{0,p}$ (which we also know is indecomposable – unlike the territory where Mashke's Theorem holds) has no nonzero element fixed by G. For general G, it is harder to compute the rank of $V_{0,p}$ as a $\mathbb{Z}/p$ vector space (the same as $K_{\text{ab},p}$ over $\mathbb{Z}_p$).

Still, [**Fr02**, Thm. 2.8] theoretically nails it, and in practice gives modules whose ranks are upper and lower bounds. When $G = A_5$, [**Fr95**, Part II] found the p-Modules $V_{0,p}$ for, respectively, $p = 2, 3$ and 5. Their respective ranks are 5, 4 and 6. For $p = 2$ it is the module of cosets on 5-Sylows of A_5. For $p = 5$ it identifies with an extension of one adjoint representation of $\text{PSL}_2(\mathbb{Z}/5) = A_5$ with another. In the p-split case the rank is given by Scheier's Thm. for a closed subgroup of a pro-free group. Still, that doesn't trivialize writing out the actual module.

At this point, one could get stuck on computing these modules for all A_n s, but that isn't the goal. Still, it is comforting to have a reservoir of groups where these modules are available, should we need examples lacking a complete theory. Thomas Weigel has provided these for all the groups $\text{PSL}_2(\mathbb{Z}/p)$ and all primes p dividing the order of this group [**Wew01**].

Modding out $K_{\text{ab},p}$ by $p^k K_{\text{ab},p}$ gives a series of finite groups, $G_{k,\text{ab}}$, $k \ge 0$, with $G_{0,\text{ab}} = G$, $G_{1,\text{ab}} = G_1$, and $\ker(G_{k+1,\text{ab}} \to G_{k,\text{ab}})$ naturally isomorphic to $V_{0.p}$ as a $\mathbb{Z}/p[G]$ module.

1.4. What is a M(odular) T(ower)? This proposal needs just the definition of what earlier papers called an *abelianized* MT, so we restrict to that case, still calling it a MT.

The first ingredient for a MT is a p-perfect group G and a collection of p' conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$ in G. This defines an infinite collection of sets – called Nielsen classes – on which the braid group on r-strings acts.

As usual, the notation $\langle g_1, \dots, g_r\rangle$ denotes the group generated by the collection of elements $g_1, \dots, g_r$. Here is the definition of the *level 0* set:

$$\mathrm{Ni}(G, \mathbf{C}) = \{\boldsymbol{g} \in G^r \mid \ \boldsymbol{g} \in \mathbf{C}, \Pi(\boldsymbol{g}) \stackrel{\text{def}}{=} g_1 \cdots g_r = 1, \text{ and } \langle \boldsymbol{g}\rangle = G\}.$$

The 1st condition means, in some order and with correct multiplicity, entries of $\boldsymbol{g}$ are in $\{\mathrm{C}_1, \dots, \mathrm{C}_r\}$. We call the three conditions, respectively, *conjugacy*, *product-one* and *generation*.

This makes sense without G being p-perfect. Yet, if $\mathbf{C}$ are p' conjugacy classes, without G being p-perfect the Nielsen classes are empty [**Fr06**, Lem. 2.1].

The level k set looks similar, but is written as $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})$, $k \ge 0$, where the $G_{k,\mathrm{ab}}$ s are from §1.3. The notation $\mathbf{C}$ here interprets the classes of $\mathbf{C}$ as in $G_{k,\mathrm{ab}}$. That is from an easy case of the Schur-Zassenhaus lemma: if $H \to G$ is a cover of groups, with a p-group kernel, then any p' conjugacy class in G lifts uniquely to a p' conjugacy class of H.

The natural homomorphism $G_{k+1,\mathrm{ab}} \to G_{k,\mathrm{ab}}$ induces a map on the corresponding Nielsen classes. Using these maps we can speak of a *projective sequence* of elements in the collection $N_{G,\mathbf{C}} = \{\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})\}_{k=0}^{\infty}$.

The *braid group* B_r, has generators $Q_1, \dots, Q_{r-1}$ that act on

$$\begin{aligned} F_r \quad &= \langle \bar\sigma_1, \dots, \bar\sigma_r\rangle (\text{free generators}) \text{ by} \\ ((\bar\sigma_1, \dots, \bar\sigma_r))Q_i \quad &\mapsto (\bar\sigma_1, \dots, \bar\sigma_{i-1}, \bar\sigma_i\bar\sigma_{i+1}\bar\sigma_i^{-1}, \bar\sigma_i, \bar\sigma_{i+2}, \dots, \bar\sigma_r), \end{aligned}$$

fixing all but ith and i+1st coordinates. All $Q \in B_r$ fix $\bar{\bar\sigma} = \bar\sigma_1 \cdots \bar\sigma_r$.

The quotient of B_r by the normal subgroup generated by

$$Q_1Q_2 \cdots Q_{r-1}Q_{r-1} \cdots Q_1$$

is the *Hurwitz monodromy group* H_r. That kernel acts as conjugations by F_r on $(\bar\sigma_1, \dots, \bar\sigma_r)$.

(1.4) So H_r acts through $F_r/\langle\bar{\bar\sigma}\rangle$ on any Nielsen class $\mathrm{Ni}(G, \mathbf{C})^*$ ($*$ indicating an equivalence on Nielsen classes).

Use small $q \in H_r$ to represent the image of capital $Q \in B_r$. The *shift* on Nielsen classes is given by $\mathbf{sh} : (g_1, \dots, g_r) \mapsto (g_2, g_3, \dots, g_r, g_1)$. The operator $q_1q_2 \cdots q_{r-1}$ represents it. For $r = 4$ (resp. $r \ge 5$), the *cusp group* is $\langle q_2, \mathbf{sh}^2, q_1q_3^{-1}\rangle \stackrel{\text{def}}{=} \mathrm{Cu}_4$ (resp. $\langle q_2\rangle = \mathrm{Cu}_r$).

Note that $H_r = \langle \mathbf{sh}, q_2\rangle$. We often restrict to where $*$ in (1.4) one of these equivalences.

(1.5a) *Inner*: $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}} = \mathrm{Ni}(G, \mathbf{C})/G$, $g \in G$ conjugates $\boldsymbol{g} \in \mathrm{Ni} \mapsto g\boldsymbol{g}g^{-1}$; or

(1.5b) *Inner reduced*: same as inner if $r \ge 5$, and for $r = 4$, $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in,rd}} = \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}/\langle \mathbf{sh}^2, q_1q_3^{-1}\rangle$.

Traditionally, the target equivalence has been *absolute* where you mod out on Nielsen classes by the normalizer, $N_{S_n}(G)$, of G in S_n. All covers have a natural associated permutation representation of degree equal to the degree of the cover. For example, [**LOs08**] uses absolute equivalence: $G = A_n$ or S_n in all their examples. Still, when $G = A_n$, the examples of §4 show why using inner equivalence, even in their cases, gives a stronger result.

Definition 1.1. A (combinatorial) MT is a *projective* system of (nonempty) H_r orbits on $N_{G,\mathbf{C}}$ (or equivalently on $N_{G,\mathbf{C}}^{\mathrm{in,rd}}$; again, suppress notation for the choice of p).

Observe: Given an element of $\mathrm{Ni}(G, \mathbf{C})$ it is a nontrivial check if above it there are Nielsen class elements that map to it. This is what is meant by deciding if (1.3c) holds. [**Fr06**, §3.1] refers to this definition of a MT (restricting to orbits) as a *component branch* of a MT. Ditto for the following geometric form of the definition.

Definition 1.2. A (geometric) MT is a projective system of (nonempty) irreducible reduced Hurwitz space components $\{\mathcal{H}'_k\}_{k=0}^{\infty}$ corresponding to a combinatorial MT.

[**BFr02**, §4.1], for example, has details on the correspondence between the combinatorial and geometric description.

I conclude by saying a little more on the geometric spaces comprising a MT. The archetypal case is any modular curve tower for a prime p. Example: Combinatorially take level 0 to be $\mathrm{Ni}(D_p, \mathbf{C}_{2^4})^{\mathrm{in,rd}}$ with p an odd prime, D_p the dihedral group of order $2p$ and $\mathbf{C}_{2^4}$ four repetitions of the involution class. There is only one braid orbit at each level. What you get geometrically is classically denoted $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$ ($X_1(p^{k+1})$ without its cusps). Level k points (over K) correspond to *K involution realizations* of $D_{p^{k+1}}$.

[**Fr06**, §6.1] explains giving *all* modular curves in this style, though the observation in this case was made long ago [**Fr78**, §2.B].

[**BFr02**] had one major goal: Establish the Main Conjecture in a serious historically motivated cases. One case there was $G = A_5$ and four 3-cycle conjugacy classes – $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})$ – because of questions asked by Serre, private to me at first, and then in print.

[**Fr08a**, Inv. Cor. 2.3] reproves briefly [**Ser90a**] as a prelude to its Main Result – in the style I showed Serre how to use the braid group. It uses [**Ser90b**] to construct "automorphic functions" on the + components of Hurwitz spaces attached to $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ (Ex. 3.2). The quotes are because the result avoids going to a universal covering space.

2. Cusps and three Frattini Principles

This section gives the combinatorial description of cusps. Then it reviews three Frattini principles from [**Fr06**]. These translate the combinatorial description into geometric cusp properties.

2.1. Cusp Types. Let $\mathcal{O}$ be a braid orbit on a reduced Nielsen class: a combinatorial Hurwitz space component. A (combinatorial) cusp is a cusp group (Cu_r, §1.4) sub-orbit in $\mathcal{O}$.

Recall the affine j-line: Unordered 4-tuples of distinct points on $\mathbb{P}^1_z$ modulo the action of $\mathrm{PGL}_2(\mathbb{C})$ (Möbius transformations). [**Ah78**, p. 282] calls this $J(\tau)$, and [**BFr02**, §2.2.2] calls it J_4, as a special case of J_r. This has a natural compactification to $\mathbb{P}^1_j$. Use a standard normalization of $\mathbb{P}^1_j$, so that $j = 0$ (resp. $j = 1$) correspond to fixed points of order 3 (resp. 2) elliptic elements in $\mathrm{PSL}_2(\mathbb{Z})$.

Then, there is a natural compactification $\bar{\mathcal{H}}(\mathcal{O})$, of $\mathcal{H}(\mathcal{O})$ that covers $\mathbb{P}^1_z$; the only ramification is over $j = 0, 1$ and ∞. The geometric cusps – corresponding to the combinatorial cusps – are the points of $\bar{\mathcal{H}}(\mathcal{O})$ over $j = \infty$ [**BFr02**, Prop. 2.3].

For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, use the notation

$$\langle g_2, g_3\rangle = H_{2,3}(\boldsymbol{g}) \text{ and } \langle g_1, g_4\rangle = H_{1,4}(\boldsymbol{g}).$$

Denote the order of $g \in G$ by $\mathrm{ord}(g)$. Here are the three cusp types [**Fr06**, §3.2]:

- *p-cusps*: $p|\mathrm{ord}(g_2g_3)$;
- *g-p' cusp*: $H_{2,3}(\boldsymbol{g})$ and $H_{1,4}(\boldsymbol{g})$ are p' groups; and
- *o(nly)-p'*: those cusps that are neither p nor g-p'.

A g-p' cusp with $\mathrm{ord}(g_2g_3) = 1$ has a shift of the form $(g, g^{-1}, g', (g')^{-1})$. A cusp represented by such a form is called an H-M, or *H(arbater)-H(umford)*, cusp.

Recent progress on MT properties comes from using cusps well. Example: Generalizing g-p' cusps is tacit in [**Fr08a**, Main Thm.] describing precisely components of $\mathcal{H}(A_n, \mathbf{C}_{3^r})$ (Ex. 3.2). Here is how g-p' cusp generalizes to $r \geq 5$, with $\mathbf{C}$ a set of p' classes of G (notation of §1.4).

DEFINITION 2.1. An r-tuple $\boldsymbol{g} = (g_1, \dots, g_r) \in \mathrm{Ni}(G, \mathbf{C})$ defines a (first order) *g-p' element* if it partitions as $(\boldsymbol{g}_1, \dots, \boldsymbol{g}_t)$ so that:

- $\langle \boldsymbol{g}_i \rangle = G_i$ is a p' group; and
- $\langle \Pi(\boldsymbol{g}_i), i = 1, \dots, t \rangle$ is also a p' group.

It is a representative of a *g-p' cusp* if the properties above are stable under the cusp group Cu_r.

You can shift any g-p' element (apply **sh** in §1.4) into a representative for a g-p' cusp. So, having a g-p' cusp is a braid orbit invariant. Refer to such a braid orbit as g-p'. The higher order (inductive definition) of g-p' cusp is in [**Fr08b**, App. B.2].

The major result about g-p' cusps is Princ. 2.4. If $\mathcal{O}$ is a g-p' braid orbit of $\mathrm{Ni}(G, \mathbf{C})$, and $H \to G$ is a cover with p-group kernel. Then, there is a g-p' orbit of $\mathrm{Ni}(H, \mathbf{C})$ over $\mathcal{O}$.

The shift of H-M representatives gives an especially simple *genre* of g-p' cusps. One reason is *universality*: They are g-p' cusps for all allowed primes p (those not dividing orders of elements in $\mathbf{C}$). One pre-liminary goal is to classify the different *genres* of universal g-p' cusps. This should be easy, since we have experience using them, say in [**FV91**, App.]. O. Cau's thesis aims to test the following (one result is in [**Cau08**]).

GOAL 2.2 (g-p' Component). *Suppose a braid orbit contains all g-p' cusps of a given universal genre. Does* [**Fr95**, Thm. 3.21] *– for shifts of H-M cusps – generalize to say the corresponding Hurwitz space component has definition field* $\mathbb{Q}$.

Modular curve towers have p and g-p' (actually, shift of H-M), but no o-p', cusps ([**Fr08b**, App. B.1] or §4.1). By contrast, the level 0 Hurwitz spaces of §4.2, where $p = 2$, have H-M cusps and o-$2'$ cusps, but no 2-cusps.

Yet, by level 1, each has several 2-cusps, even though some o-$2'$ cusps persist here. Still, for $n \equiv 5 \mod 8$, the cusp structure by level 1 and above looks much like that of modular curves from the existence of a spire (as in (1.1c)).

2.2. Review of the Three Frattini Principles. [**Fr06**, Prop. 3.3] reduces the Main Conjecture on MT for any p-perfect group to where G is centerless. As a corollary of [**BFr02**, Prop. 3.21], if G is centerless, then so are the characteristic groups $\{G_{k,\mathrm{ab}}\}_{k=0}^{\infty}$. This hypothesis is significant for many technical points, but one practical implication is that the non-reduced Hurwitz spaces have fine moduli.

2.2.1. *Frattini Principle 1: FP1.* Assume we have *inner*, reduced Nielsen classes (§1.4). Given the center of G is p', the first Frattini Principle says this. The p order of any level k combinatorial p-cusp is the p ramification index (over the j-line) of the corresponding geometric cusp. Further, it is a Frattini property that any (level $k+1$) cusp above it has index one more power of p. So, once you have a p-cusp, the p index of cusps above it grows with the levels.

This is in agreement with what happens for the modular curves $X_1(p^{k+1})$. A cusp of width p^u, $u \geq 1$, has above it at level $k+1$ only cusps of width p^{u+1} over it. The combinatorial generalization to MT cusps considers any projective system of cusp representatives:

$$\{{}_k\boldsymbol{g} = ({}_kg_1, {}_kg_2, {}_kg_3, {}_kg_4) \in \mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})\}_{k=0}^{\infty}.$$

PRINCIPLE 2.3 (Frat. Prin. 1). $p^u || \mathrm{ord}({}_kg_2{}_kg_3)$*,* $u \geq 1$ *if and only if* $p^{u+1} || \mathrm{ord}({}_{k+1}g_2{}_{k+1}g_3)$.

From FP1, [**Fr06**, Thm. 5.1] lists those MTs, with $r = 4$, that could fail the Main Conjecture: For $k >> 0$ the relative covers $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$ are either unramified (covers of genus 1 curves) or, as covers, they are equivalent to degree p rational function maps (either polynomials or Redyi functions). Also, if these (reduced) spaces have fine moduli (a stronger condition than for the ordinary spaces), then they must be Redyi functions.

So, for $r = 4$, [**Fr06**, Thm. 5.1] essentially reduced an explicit approach to the Main Conjecture to producing p cusps. If you have them, the genus of MT levels grows quickly. This is a stronger result than [**CadT08**], though it has mainly been shown for the *pure-cycle* Nielsen classes of genus 0 covers considered in [**Fr08b**] (see §3.1).

2.2.2. *Frattini Principle 2: FP2.* Given a g-p' cusp, [**Fr06**, Princ. 3.6 — Frat. Princ. 2] effectively constructs a projective sequence of Modular Tower components. It does so by constructing a projective sequence of g-p' cusp orbits: by forming a g-p' cusp branch.

Principle 2.4 (Frat. Prin. 2). *If* ${}_0\boldsymbol{g} \in \mathrm{Ni}(G_0, \mathbf{C})$ *represents a g-p' cusp, then above it there is a g-p' cusp* $\{{}_1\boldsymbol{g} \in \mathrm{Ni}(G_1, \mathbf{C})\}$. *Further, there is a projective system of g-p' cusps on any* MT *over the component defined by* ${}_0\boldsymbol{g}$ *(either the abelianized* MT *of §1.4, or the complete* MT *of* [**Fr95**]*).*

Combining FP1 and FP2 makes sense. The existence of p-cusps bodes well for the dimension of the sheaf of holomorphic differentials to grow, with the levels, even when the reduced Hurwitz space has dimension exceeding 1. So, it helps approach the Main Conjecture for all values of r.

Goal 2.5 (Kodaira dimension). *On a* MT *defined by a g-p' cusp, relate the existence of p-cusps to the Kodaira dimension of the levels.*

2.2.3. *Frattini Principle 3:* FP3. Let $\mathrm{Ni}(G, \mathbf{C})$ be a Nielsen class, $H \to G$ a *central* Frattini cover with p kernel, and $\mathbf{C}$ some p' conjugacy classes. Then, for $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ there is a unique $\tilde{\boldsymbol{g}} \in H^r$ over $\boldsymbol{g}$, lying in the lifted p' classes of $\mathbf{C}$. Define the *lifting* invariant of $\boldsymbol{g}$ to be

$$s_{H/G} \overset{\text{def}}{=} \prod_{i=1}^{r} \tilde{g}_i \in \ker(H \to G). \tag{2.1}$$

This is independent of the choice of $\boldsymbol{g}$ in its braid orbit [**Fr95**, §III.D].

Special case defined by Serre [**Ser90a**, Braid Arg. of Fried]: Spin_n^+ is the (unique) nonsplit degree 2 cover of the connected component O_n^+ (of I_n) of orthogonal group. Regard S_n as $< O_n$ (orthogonal group); $A_n < O_n^+$, kernel of the determinant map. Then, Spin_n is the pullback of A_n to Spin_n^+: $\ker(\mathrm{Spin}_n \to A_n) = \{\pm 1\}$.

You can apply this lifting invariant to any Nielsen classes of odd-order elements, and then we call it the spin-lifting invariant.

With $r = 4$, assume $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$. Let $R_G \to G$ be the central extension of G with $\ker(R_G \to G)$ the maximal quotient of M_0 on which G acts trivially (§1.3). That is: $\ker(R_G \to G)$ is the maximal exponent p quotient of the Schur multiplier of G. Use notation as in §2.1 with

$$H_{2,3}(\boldsymbol{g}) = H_{2,3} \text{ and } H_{1,4}(\boldsymbol{g}) = H_{1,4},$$

replacing G. [**Fr06**, Lem. 4.15] says this gives a natural map $\beta_{2,3} : R_{H_{2,3}(\boldsymbol{g})} \to R_G$. Denote the generalization of the lifting invariant of $\boldsymbol{g}$ for $R \to G$ in §1.3 by $s_{G,p}(\boldsymbol{g})$. Thm. 5.5 uses this result.

Principle 2.6 (o-p' cusps). *If $\boldsymbol{g}$ is an o-p' cusp (§2.1), then* [**Fr06**, Princ. 4.24]*:*

$$s_{G,p}(\boldsymbol{g}) = \beta_{1,4}(s_{H_{1,4},p}((g_4 g_1)^{-1}, g_4, g_1))\beta_{2,3}(s_{H_{2,3},p}(g_2, g_3, (g_2 g_3)^{-1})).$$

3. Some Known Connectedness Results

§3.1 lists five examples of connectedness results related to this proposal. §3.2 discusses the most overriding of these results for comparing [**Fr08a**, Thms. 1.2 and 1.3] and [**LOs08**].

3.1. Five examples.
Use the Nielsen class notation of §1.4.

Take $G = D_{p^{k+1}}$ with p odd, and $\mathbf{C}_2^*$, the conjugacy class of an involution. Consider the index set of even positive integers, $I = 2\mathbb{N}$. There is one braid orbit on $\mathrm{Ni}(G, \mathbf{C}_{2^r}^*)$, $r \in I$, and the corresponding Hurwitz space $\mathcal{H}_r^{\mathrm{in,rd}}$ is the space of cyclic p^{k+1} covers of hyperelliptic jacobians of genus $\frac{r-2}{2}$ [**DFr94**, §5]. §4.1 reminds of the classical computation for $r = 4$. Since it presents the space as an upper half-plane quotient, it is clearly connected.

We now compare the case above with 4 other cases. Here C_d corresponds to the *pure-cycle* conjugacy class of length d: One disjoint cycle of length d in S_n, the rest of length 1.

(3.1a) One braid orbit on $\mathrm{Ni}(S_n, \mathbf{C}_{2^r})$, defined by $r \in I$ elements in C_2: Clebsch, 1872, used for connectedness of genus g moduli.
(3.1b) One (resp. two) braid orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})^*$ with $r = n-1$ (resp. $r \geq n$), $* =$ in or abs equivalence: [**Fr08a**, Thms. 1.2,1.3], see Ex. 3.2.
(3.1c) Absolute spaces of pure-cycle covers $\mathbf{C}_{d_1 \cdots d_r}$ in S_n ($G = A_n$ if all d_i odd, otherwise S_n) of genus 0 ($\sum_{i=1}^r d_i - 1 = 2(n-1)$) [**LOs08**, Thm. 5.5].
(3.1d) Hurwitz spaces for Nielsen classes with all conjugacy classes appearing sufficiently often: CFPV, 1991, see §3.2.

When $n = 4$ (or 3) there are two conjugacy classes of 3-cycles, so $\mathbf{C}_{3^r}$ is ambiguous. Similarly, for other odd pure-cycle cases where there is an n or $n-1$-cycle in A_n.

Describing level 0 of the $(A_n, \mathbf{C}_{3^r}, p = 2)$ MT components generalizes Serre's Stiefel-Whitney approach to Spin covers [**Ser90b**]. The case $* =$ abs, and $r = n-1$ in (3.1b) overlaps with (3.1c). For the case $r \geq n$ in (3.1b), the covers in the family have genus $r-(n-1)$.

Given an absolute Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ and any braid orbit $\mathcal{O}^{\mathrm{abs}}$, we may always consider a braid orbit $\mathcal{O}^{\mathrm{in}}$ of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ above it. Then, there is a map between the corresponding Hurwitz spaces $\Psi_{\mathcal{O}^{\mathrm{in}}, \mathcal{O}^{\mathrm{abs}}} : \mathcal{H}(\mathcal{O}^{\mathrm{in}}) \to \mathcal{H}(\mathcal{O}^{\mathrm{abs}})$. [**FV91**, Main Thm.] says $\Psi_{\mathcal{O}^{\mathrm{in}}, \mathcal{O}^{\mathrm{abs}}}$ is a Galois cover with group a subgroup of the automorphism group of G in S_n modulo G. So, this map has degree 2 or 1 in each of the (3.1c) cases where $G = A_n$. As §4.2 examples indicate, both cases occur.

3.2. Conway-Fried-Parker-Völklein Thm. Consider regular realizations of a group G over $\mathbb{Q}$. For there to be any such realizations, there must be a Hurwitz space component over $\mathbb{Q}$. Further, the *Branch Cycle Lemma* ([**Fr77**, p. 62], but recounted in many places, see top of §1) says this requires $\mathbf{C}$ is a *rational union* (as a set with multiplicity, it is closed under powers prime to $|G|$). An addition to [**FV91**] says this [**Fr08a**, App. E.2].

Theorem 3.1 (Branch-Generation Thm.). *Assume G centerless and $\mathbf{C}^*$ a rational union of distinct (nontrivial) classes in G whose elements generate G. Then, an infinite set $I_{G,\mathbf{C}^*}$ indexes distinct absolutely irreducible $\mathbb{Q}$ varieties $\mathcal{R}_{G,\mathbf{C}^*} \stackrel{\mathrm{def}}{=} \mathcal{R}_{G,\mathbf{C}^*,\mathbb{Q}} = \{\mathcal{H}_i\}_{i \in I_{G,\mathbf{C}^*}}$ with:*

(3.2a) *a finite-one map $i \in I_{G,\mathbf{C}^*} \mapsto {}_i\mathbf{C}$, r_i conjugacy classes of G supported in $\mathbf{C}^*$; and*
(3.2b) *the RIGP holds for G with conjugacy classes $\mathbf{C}$ supported in $\mathbf{C}^*$ $\Leftrightarrow$ $i \in I_{G,\mathbf{C}^*}$ with $\mathbf{C} = {}_i\mathbf{C}$ and $\mathcal{H}_i$ has a $\mathbb{Q}$ point.*

The emphasis is on $|I_{G,\mathbf{C}^*}| = \infty$. Unlike (3.1d), we don't need all conjugacy classes. Ex. 3.2 fits this rubric, but precisely. Rather than just $|I_{G,\mathbf{C}^*}| = \infty$, it describes $I_{G,\mathbf{C}^*}$ exactly. For general $(G, \mathbf{C}^*)$, Thm. 3.1 still requires unknown repetitions in ${}_i\mathbf{C}$ of conjugacy classes from $\mathbf{C}^*$.

Example 3.2 (3-cycle cases). From [**Fr08a**, p. 2]: If $G = A_n$, $\mathbf{C}^* = \{\mathrm{C}_3\}$, then $i \mapsto \mathbf{C}_{3^{r_i}}$ with $r_i \geq n$ is two-one. Denote indices mapping to r by $i_r^{\pm}$. Covers in $\mathcal{H}_{i_r^{\pm}}$ are Galois closures of degree n covers $\varphi : X \to \mathbb{P}^1_z$ with 3-cycles for local monodromy. The $\pm$ correspond to spin lifting values in §2.2.3: A cover has a branch cycle description with $+1$ spin lifting invariant if and only if it is in $\mathcal{H}_{i_r^+}$, a special case of Invariance Thm. 5.1. For $r_i = n-1$ the map $i \mapsto \mathbf{C}_{3^{r_i}}$ is one-one.

Also, only $\mathcal{H}_{i_r^+}$ is level 0 of a MT for $p = 2$: Princ. 2.4 (resp. Thm. 5.2) for why $\mathcal{H}_{i_r^+}$ is (resp. $\mathcal{H}_{i_r^-}$ is not). Still, we don't know if such a MT is over a number field (satisfies (1.3a)).

Goal 3.3. *Establish if it is possible to be as precise on all pure-cycle Nielsen classes as given by the results* (3.1b). *In all pure-cycle Nielsen classes where $G = A_n$ (dropping the genus 0 condition of* (3.1c)*), establish which components are the analog of the + components of Ex. 3.2.*

Results (3.1a) (1 component) and (3.1b) (usually two components) look very different, don't they? The former has short easy proofs. Yet, there is a surprise relation between them.

There is a central Frattini extension $R \to S_n$ of degree 4 (say, [**Ser92**, §9.1.3]). Let g' be a lift of a 2-cycle in S_n to R. Denote its conjugacy class in R by C'. If $\mathrm{C}' \to \mathrm{C}_2$ maps 1-1, then (2.1) extends to define $s_{R/S_n}(\boldsymbol{g})$ for any $\boldsymbol{g}$ in a (3.1a) Nielsen class. So, for certain, as in (3.1b), there would be 2 or 4 connected components in (3.1a) for large r values, not 1.

The file CFPV.html in [**Fr08d**] shows directly why $\mathrm{C}' \to \mathrm{C}$ is a 4-1 mapping, and how the 2-cycle and 3-cycle cases together faithfully guide why we know exactly what are the $\mathbb{Q}$ components in Thm. 3.1; so long as there are many repetitions of the conjugacy classes in $\mathbf{C}^*$. It is this "many repetitions" business that has caused anxiety among those Hurwitz-space-aware people who would like to use these results. This adds to the significance of establishing Goal 3.3.

4. sh-incidence matrices of Nielsen classes

The *width* of a cusp means the ramification index of the cusp over $j = \infty$. The **sh**-incidence matrix is a pairing on cusps (as Cu_r orbits of reduced Nielsen classes) [**BFr02**, §2.10]:

$$({}_{\mathbf{c}}O, {}_{\mathbf{c}}O') \mapsto |{}_{\mathbf{c}}O \cap ({}_{\mathbf{c}}O')\mathbf{sh}| \text{ (apply } \mathbf{sh} \text{ to each element in } {}_{\mathbf{c}}O').$$

Two basic lemmas hint at the extra structure revealed by it.

(4.1a) Blocks of the matrix correspond to components of the Hurwitz space [**BFr02**, Lem. 2.26].

(4.1b) Elliptic fixed points (ramification over $j = 0$ or 1) contribute to the diagonal [**Fr08b**, Lem. 4.8].

This matrix organizes braid orbits since cusp orbits are easy to compute. Our computations don't use a computer program (like **GAP**). Still, I suggested to Kay Maagard the braid package [**MShStV**] (for Nielsen class orbits) would benefit from a sub-routine on the **sh**-incidence matrix. Known lemmas should make it efficient.

sh-incidence makes sense for all $r \geq 4$, and it applies to all *reduced* Nielsen classes. When $r = 4$, the matrix is *symmetric* [**BFr02**, §2.10].

§4.1 gives this matrix for the modular curve $X_0(p^2)$ (p odd as in §1.3). §4.2 does it for the unique component of the Hurwitz space with Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, $n \equiv 5 \mod 8$. In both cases there is just one matrix block, so there is just one Hurwitz space component in these inner Nielsen classes. §4.3 gives it for a case of [**Fr08a**] not included in [**LOs08**] because the Hurwitz space is a family of genus 1 curves.

4.1. Level 1 of modular curves.

[**Fr08b**, §B.1] computes the $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{*,\mathrm{rd}}$ sh-incidence matrix for both absolute ($* = \mathrm{abs}$) and inner ($* = \mathrm{in}$) classes. Table 1 gives the absolute case.

TABLE 1. **sh**-incidence for $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$ (level $k = 1$)

Cusp orbit	${}_{\mathbf{c}}O_{p^2}$	${}_{\mathbf{c}}O_{a,p}, a \in (\mathbb{Z}/p)^*$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_{p^2}$	$p(p-1)$	1	1
${}_{\mathbf{c}}O_{a,p}, a \in (\mathbb{Z}/p)^*$	1	0	0
${}_{\mathbf{c}}O_1$	1	0	0

Row 1 $\leftrightarrow$ width p^2: H-M rep. cusp (§2.1), ${}_{\mathbf{c}}O_{p^2}$;
Rows for $a \in (\mathbb{Z}/p)^* \leftrightarrow$ cusps ${}_{\mathbf{c}}O_{a,p}$, of width 1; and
Last row $\leftrightarrow$ width 1 cusp ${}_{\mathbf{c}}O_1$: shift of the H-M rep.

Since this is an absolute, not inner, Nielsen class case, FP1 (§2.2.1) may not apply. Indeed, it doesn't for the width 1 cusps corresponding to $a \in (\mathbb{Z}/p)^*$. These are p-cusps. Yet, as j-line covers their ramification is not divisible by p. All cusps, however, above them at level 2 (on

$X_0(p^3)$) do have ramification order p, and the ramification goes up by a multiple of p every level after that. That is, at high enough levels, FP1 applies even in the absolute case.

The argument of [**Fr08b**, App. B.1] is shorter, with more cusp information from the sh-incidence matrix, than the classical argument for listing modular curve cusps (say, [**Sh71**, §1.6]).

4.2. sh-incidence for $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$**,** $n \equiv 1 \mod 4$**.** Liu-Osserman (3.1b) says there is one braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs}}$, the absolute Nielsen classes (mod out by S_n on 4-tuples). Using cusps makes quick work of this [**Fr08b**, Table 1]. Table 2 gives the **sh**-incidence matrix for $n = 5$. That reveals the natural pattern in all $n \equiv 1 \mod 4$ absolute cases: There is one cusp of each odd width between n and 1; all 2's above the subdiagonal; all 1's along it, and all 0's below it [**Fr08b**, Prop. 5.1].

TABLE 2. **sh**-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{3^4}^{\mathrm{in,abs}}$

Cusp orbit	${}_{\mathbf{c}}O_5$	${}_{\mathbf{c}}O_3$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_5$	2	2	1
${}_{\mathbf{c}}O_3$	2	1	0
${}_{\mathbf{c}}O_1$	1	0	0

One way to see the inner result is stronger is to look at $n \equiv 1 \mod 8$. Then, there are two braid orbits, so two Hurwitz space components. Indeed, these two components are conjugate over a quadratic extension of $\mathbb{Q}$ [**Fr08b**, Prop. 6.3].

By contrast, when $n \equiv 5 \mod 8$ the inner spaces have one component [**Fr08b**, Prop. 5.15]. You see this for $n = 13$ because there is one block in Table 3 (see (4.1)). Here is a description of these cusps using a parameter ℓ running over odd integers between n and 1 [**Fr08b**, Cor. 5.9].

TABLE 3. Abs-inn cusp form for $n = 13$

Cusp orbit	${}_{\mathbf{c}}O_{13}$	${}_{\mathbf{c}}O_{11}$	${}_{\mathbf{c}}O_9$	${}_{\mathbf{c}}O_7$	${}_{\mathbf{c}}O_5$	${}_{\mathbf{c}}O_3$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_{13}$	$\frac{0\vert 2}{2\vert 0}$	$\frac{1\vert 1}{1\vert 1}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{0\vert 2}{2\vert 0}$	$\frac{1\vert 1}{1\vert 1}$	$\frac{1}{1}$
${}_{\mathbf{c}}O_{11}$	$\frac{1\vert 1}{1\vert 1}$	$\frac{0\vert 2}{2\vert 0}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{1\vert 1}{1\vert 1}$	$\frac{0\vert 1}{1\vert 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_9$	$2\vert 2$	$2\vert 2$	4^0	4	$1\vert 1$	$0\vert 0$	0
${}_{\mathbf{c}}O_7$	$2\vert 2$	$2\vert 2$	4	2^1	$0\vert 0$	$0\vert 0$	0
${}_{\mathbf{c}}O_5$	$\frac{0\vert 2}{2\vert 0}$	$\frac{1\vert 1}{1\vert 1}$	$\frac{1}{1}$	$\frac{0}{0}$	$\frac{0\vert 0}{0\vert 0}$	$\frac{0\vert 0}{0\vert 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_3$	$\frac{1\vert 1}{1\vert 1}$	$\frac{0\vert 1}{1\vert 0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0\vert 0}{0\vert 0}$	$\frac{0\vert 0}{0\vert 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_1$	$1\vert 1$	$0\vert 0$	0	0	$0\vert 0$	$0\vert 0$	0

(4.2a) There are two cusps of width ℓ if $\ell \equiv \pm 3, \pm 5 \mod 8$ – corresponding to notation for two columns (or rows) under (to the right of) ${}_{\mathbf{c}}O_\ell$;
(4.2b) for $\ell \equiv \pm 1 \mod 8$, there is one cusp, of width 2ℓ;
(4.2c) none of the cusps are 2-cusps;
(4.2d) adding entries down any column for a cusp sums to the width; and
(4.2e) the unique fixed point of γ_0 (resp. γ_1) is indicated by the 0 (resp. 1) superscript in the diagonal $({}_{\mathbf{c}}O_9, {}_{\mathbf{c}}O_9)$ (resp. $({}_{\mathbf{c}}O_7, {}_{\mathbf{c}}O_7)$) position.

Adding the fixed points of $\gamma_0 = q_1 q_2$ and $\gamma_1 = \mathbf{sh}$ gives the genuses of the reduced spaces as recounted for $n \equiv 1 \mod 4$ [**Fr08b**, Prop. 5.15]. They genus grows quadratically in n.

4.3. sh-incidence for a space of genus 1 curves. Let $C_{\pm 3}$ be the two conjugacy classes of 3-cycles in A_4. Then, there are two components of the level 0 MT for $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$, corresponding to two braid orbits Ni_0^+ and Ni_0^- on $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$. Here is the idea.

There are three elements (labeled [1], [2], [3]) in $\mathrm{Ni}(A_3, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$. Label Nielsen elements in $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ over, say [1], as $\boldsymbol{g}_{1,i}$, $i = 1, \dots, 5$.

We find $\boldsymbol{g}_{1,1}, \boldsymbol{g}_{1,2}, \boldsymbol{g}_{1,3}$ over [1] are permuted as a set by **sh**. They map by q_2 respectively to $\boldsymbol{g}_{2,1}, \boldsymbol{g}_{2,2}, \boldsymbol{g}_{2,3}$ over [2]. Similar action by **sh** and q_2 give a listing of elements $\boldsymbol{g}_{j,i}$, $i = 1, \dots, 5$ and $j = 1, 2, 3$, over $[j]$. There are six total q_2 reduced orbits, labeled $O_{1,1}$, $O_{1,3}$, $O_{3,1}$, $O_{1,4}$, $O_{3,4}$ and $O_{3,5}$. Here, there are two blocks (either absolute or inner equivalence) revealing the components Ni_0^+ and Ni_0^-. Table 4 has just the block for Ni_0^+. This corresponds to the braid orbit containing H-M reps. (§2.1). So, elements in this orbit have spin-lifting invariant +1 (§2.2.3). The other orbit Ni_0^- consists of elements with spin-lifting invariant -1. Above Ni_0^- there is no MT for $p = 2$.

TABLE 4. The + component of $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$

Orbit	$O_{1,1}$	$O_{1,3}$	$O_{3,1}$
$O_{1,1}$	1	1	2
$O_{1,3}$	1	0	1
$O_{3,1}$	2	1	0

[**Fr06**, §6.4] uses these (and [**Wo64**]) to show neither of Ni_0^+ nor Ni_0^- is a modular curve.

5. Higher MT levels

[**BFr02**]'s strategy was to use two examples as a model for what to expect of a non-modular curve MT: $\mathrm{Ni}(A_5, \mathbf{C}_{3^4}, p = 2)^{\mathrm{in,rd}}$ (the first case of §4.2) and $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2}, p = 2)^{\mathrm{in,rd}}$ (§4.3). I was assiduous in assuring everything had a proof (not a **GAP** computation) – that is, basing nothing on conjectures or pure raw data.

My goal: Given support for the project in the mathematical community, I was prepared to produce towers for serious problems. [**Fr06**, §6] lists precise – non-obvious – applications to the Inverse Galois Problem, Serre's Open Image Theorem, properties of degree 0 divisors supported on cusps (think Drinfeld-Manin's result for modular curves) – based on just the examples I mentioned above. What remains of this proposal says how MTs with Liu-Osserman Nielsen classes at level 0 provide the chance to carry on those projects.

5.1. Using [BFr02] as a model. [**BFr02**, §8.5] has the sh-incidence matrix for the level 1 Nielsen class $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{3^4}, p = 2)^{\mathrm{in,rd}}$ (notation of §1.4). There are two components, $\mathcal{H}_{1,+}$ and $\mathcal{H}_{1,-}$, the former having a compactification of genus 12 and the latter genus 9. Only $\mathcal{H}_{1,+}$ is level 1 of a non-empty MT. The precise computation for this, and the analysis of real points all the way up the tower involved much of the theory developed in [**BFr02**]. Examples:

(5.1a) Level 1 real points form one connected component, on $\mathcal{H}_{1,+}$ [**BFr02**, §8.6].

(5.1b) The Schur multiplier of $G_1(A_5)$ is $\mathbb{Z}/2$ and the (2.1) lifting invariant is exactly what separates the components $\mathcal{H}_{1,\pm}$.

5.1.1. *More on the Spin-lifting invariant.* Here is a brief history of Invariance Thm. 5.1. Serre asked what I thought was the formula for $n-1$ 3-cycles in A_n. My answer: $s_{\mathrm{Spin}_n/A_n} = (-1)^{n-1}$. He generalized it to $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C})$ whenever $\mathbf{C}$ consists of odd order conjugacy classes, so long as covers in this Nielsen class have genus 0 [**Ser90a**].

It is trivial to revert to the case of pure-cycles and the notation $d_1, \dots, d_r$ of (3.1c) for the odd pure-cycle lengths. [**Fr08a**, Inv. Cor. 2.3] shows it quickly from the 3-cycle case (Ex. 3.2).

THEOREM 5.1 (Invariance). *For $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{d_1\cdots d_r})$ satisfying, $\sum_{i=1}^r d_i - 1 = 2(n-1)$ – genus 0 condition – the spin-lifting invariant is $(-1)^{\sum_{i=1}^r \frac{d_i^2-1}{8}}$.*

The following remarks contribute to Goal 3.3. The proof of [**Fr08a**, Thm. 1.3] combined with [**BFr02**, §6] gives an algorithm to compute $s_{\mathrm{Spin}_n/A_n}(\boldsymbol{g})$ for $\boldsymbol{g}$ any pure-cycle Nielsen class representative: *without* the genus 0 condition.

With the genus 0 condition it is easy to show all Nielsen classes with odd pure-cycles are non-empty. Without genus 0, it is harder to decide whether Nielsen classes are non-empty and which values you get of the spin-lifing invariant.

The remainder of this subsection lists corollaries (albeit, not immediate) from Thm. 5.1. For any braid orbit $\mathcal{O}$ of any Nielsen class $\mathrm{Ni}(G, \mathbf{C})$, if the p-part of the Schur multiplier of G is trivial, then there is always at least one MT over $\mathcal{O}$ for the prime p [**FrK97**, Lift Princ.].

Suppose G has an embedding in A_N for some N, so that the pullback of G to Spin_N doesn't split over G. Then, $\mathbb{Z}/2$ is a quotient of the 2-part of the Schur multiplier of G.

THEOREM 5.2. *Assume further, that $\mathbb{Z}/2$ is the full quotient of the 2-part of the Schur multiplier of G and $\mathbf{C}$ has odd order branch cycles. Then, for $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ there is a* MT *for the prime 2 over the braid orbit of $\boldsymbol{g}$ if and only if $s_{\mathrm{Spin}_N/A_N}(\boldsymbol{g}) = 1$.*

Sufficiency of $s_{\mathrm{Spin}_N/A_N}(\boldsymbol{g}) = 1$ for existence of a MT and in other applications is a deeper consequence (from Thm. 5.4). For that I had Thomas Weigel's help, as recounted in §5.2.1.

[**Fr02**, §2] has a classification of Schur multipiers appropriate for applying to MTs. All the groups $G_{k,\mathrm{ab}}(A_n)$ have Schur multiplier $\mathbb{Z}/2$, the case that [**Fr02**, §2] calls *antecedent* (from Spin_n). The computation of the -1 lifting invariant for Nielsen class orbit corresponding to $\mathcal{H}_{1,-}$ is a special case of Thm. 5.2. This comes from finding an embedding of $G_1(A_5)$ in appropriate A_N s ($N = 120$ for example, [**BFr02**, Prop. 9.14]).

We also know – in far more generality than I'm stating it here [**Fr06**,] – that if the lifting invariant (2.1) separates two Nielsen class braid orbits for the group $G_{k',\mathrm{ab}}(A_n)$, then it separates two such orbits at all MT levels higher than k'. That makes sense of refering to the union of level k components corresponding to +1 lifting invariant in $\mathrm{Ni}(G_{k,\mathrm{ab}}(A_5), \mathbf{C}_{3^4})$ as $\mathcal{H}_{k,+,5,p=2}$. So, a successful conclusion to the following goal would give an example of (1.3a); the kind of connectedness result sought in $\mathbf{Goal}_4$ (end of §1.1).

GOAL 5.3. *Is there a bound on the number of components of $\mathcal{H}_{k,+,5,p=2}$ independent of k?*

Princ. 5.6 produces many high width 2-cusps as we go up all MTs with level 0 in §4.2 for $n \equiv 5 \mod 8$. This bodes well for a positive conclusion to the analog of Goal 5.3 for all those n.

5.2. p-Poincaré duality and Serre's OIT. This last subsection sketches the heavier tools that are part of this proposal.

5.2.1. *Characterizing the lifting invariant.* The phrase *dimension 2 p-Poincaré duality* [**We05**, (5.8)] expresses an exact cohomology pairing

$$H^k(M_{\boldsymbol{g}}, U^*) \times H^{2-k}(M_{\boldsymbol{g}}, U) \to \mathbb{Q}_p/\mathbb{Z}_p \stackrel{\mathrm{def}}{=} I_{G,p}. \tag{5.2}$$

Here U is any abelian p-power group that is also a $\Gamma = M_{\boldsymbol{g}}$ module, U^* is its dual with respect to $I_{G,p}$ and k is any integer. [**Ser91**, I.4.5] has the same definition, except his $M_{\boldsymbol{g}}$ is a pro-p-group, while our $M_{\boldsymbol{g}}$ is p-perfect.

Consider a braid orbit $\mathcal{O}$ in a Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ with p' classes $\mathbf{C}$, $\boldsymbol{g} \in \mathcal{O}$. Here is where $M_{\boldsymbol{g}}$ comes from. Let $\varphi : X \to \mathbb{P}^1_z$ be a cover with branch cycles $\boldsymbol{g}$. Then, $M_{\boldsymbol{g}}$ is canonically, up to *braid* isomorphism, an extension of G by the pro-p completion of the fundamental group of X. Weigel showed (5.2) holds for our $M_{\boldsymbol{g}}$. [**Fr06**, Cor. 4.19] used it to show the following.

Let $R_p \to G$ be the maximal central p-Frattini extension of G, and $s_{R_p/G}(\mathcal{O})$ as in (2.1).

THEOREM 5.4. *Then, there is a* MT *over* $\mathcal{O}$ *if and only if* $s_{R_p/G}(\mathcal{O}) = 0$.

An application of Princ. 2.6 then gives a similar result, but this time about p-cusps. For simplicity take $r = 4$, and use the notation of §2.1

THEOREM 5.5. *Let* $\boldsymbol{g} = (g_1, g_2, g_3, g_4)$ *be a representative of a g-*p' *cusp* ${}_{\mathbf{c}}O$ *in* $\mathrm{Ni}(G, \mathbf{C})$*, (so* $p \not| \mathrm{ord} g_2 g_3$*). Then, all cusps at level 1 above* ${}_{\mathbf{c}}O$ *are p-cusps if and only if* $s_{R_p/G}$ *applied to the 3-tuple* (g_1, g_2g_3, g_4) *is* $\neq 1$.

In particular, consider the Liu-Osserman examples of §4.2 with $n \equiv 5 \mod 8$*. For those, all level 1 cusps above the cusps of width* $\ell \equiv \pm 3, \pm 5 \mod 8$ *are 2-cusps.*

5.2.2. *Enhancing the modular curve-like look of a* MT. Consider the the Liu-Osserman examples of §4.2 with $n \equiv 1 \mod 8$. It's 2-cusps at level 1 differ markedly from those described in Thm. 5.5, epitimized by Princ. 5.6 – a very special case of [**Fr08b**, Cor. 5.17]. Both have H-M reps. $\boldsymbol{g}_{\text{H-M},0}$ (§2.1) at level 0. By Princ.2.4, we may consider a projective system of braid orbits $\{\mathcal{O}_k\}_{k=0}^{\infty}$ given by the shifts of H-M reps, $\tilde{\boldsymbol{g}} = \{(\boldsymbol{g}_{\text{H-M},k})\mathbf{sh}\}_{k=0}^{\infty}$ at each level.

PRINCIPLE 5.6 (Spire Princ.). *Suppose for some* k', $\boldsymbol{g}_{\text{H-M},k}$ *is a p-cusp. Then, the cusp tree of the* MT *defined by* $\tilde{\boldsymbol{g}}$ *contains a subtree — a spire (starting at level* k'*) — isomorphic to the cusp tree of a modular curve tower.*

In particular, this is true in the case of $n \equiv 5 \mod 8$ *in §4.2.*

Table 5 represents how p-cusps grow, in one sub-braid orbit, from any H-M cusp in Princ. 5.6. If $k' = 1$ in Princ. 5.6, each level ($k \geq 1$) has a cusp (lying over a non-p-cusp) of width divisible by exactly one power of p. The $\bullet$ subscripts indicate the power of p dividing the cusp width.

TABLE 5. Growth of p cusps with level

Level 1 :	$\bullet_p$			
Level 2 :	$\bullet_{p^2}$	$\bullet_p$		
Level 3 :	$\bullet_{p^3}$	$\bullet_{p^2}$	$\bullet_p$	
$\cdots$:	$\ldots$	$\ldots$	$\ldots$	$\ldots$

GOAL 5.7. *Generalize the mechanics of Princ. 5.6 to characterize existence of a spire.*

To get Hecke operators, and a full Serre Open Image Theorem, we must involve (almost) all primes. For MTs here, there are only finitely the primes dividing $|G|$. [**Fr06**, §6.1] generalized this to a *higher rank* MT. For this, let a finite group H act faithfully on a $\mathbb{Z}$ lattice L, with conjugacy classes $\mathbf{C}$ in H. Then, consider p, and $G_p = L/pL \times^s H$. We need a p-perfect assumption, etc.

Examples in this proposal suggestion, for a fixed n, taking $H = A_n$ and the standard, irreducible $n-1$ dimensional lattice L on which A_n acts [**Fr08b**, §6]. [**Fr06**, §6.3] develops the nontrivial case for $n = 3$, and conjectures precisely what should be the "complex multiplication" and the "GL_2 part" j-values for the analog of Serre's OIT.

Berger in [**Be97**], under me as a post-doc, did an analog of "complex multiplication" j-values (which, by the conjecture, should only occur when $p \equiv 1 \mod 3$).

GOAL 5.8. *Hecke correspondences for these cases should be up on my web site soon – I have a draft. The full OIT goals will take some time, starting with properly modifying Berger's analysis.*