# Asymptotic and Numerical Results for Blowing-Up Solutions to Semilinear Heat Equations

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ABSTRACT. Asymptotic and numerical results for blowing-up solutions to a class of semilinear heat equations are presented. They are obtained using the original space-time coordinate frame without any dynamical rescaling of space, time or solution. A new equivalent equation is derived using a nonlinear transformation. Under some assumptions, the asymptotic form, the time and the position of the singularity are determined. Blow-up in the original equation is equivalent to vanishing in the new equation. As a consequence of the transformation *and* the asymptotic form of the blow-up, all terms in the new equation are bounded. Thus, numerical solutions are easy to obtain right up to the blow-up time without any loss of resolution. Inverting the transformation yields the solution to the original problem.

## 1. Introduction

The study of solutions of partial differential equations which blow up is an important and challenging problem. Although physically, we might expect solutions to remain smooth and bounded, it is not uncommon to see physical quantities vary by several orders of magnitude over relatively short times. Therefore, we can think of them as nearly blowing-up. This is particularly true in combustion problems where semilinear heat equations are useful models (see [1]). A detailed study of blow-up in the mathematical model may then give some insight into the near blow-up of physical quantities and may also provide matching conditions for other, more realistic models that may be used near the blow-up time.

We shall present asymptotic and numerical results for blowing-up solutions to a class of semilinear heat equations. The results are obtained without any dynamical rescaling of space, time or solution, but are obtained using the original space-time coordinates instead. A new equivalent equation is derived by using a nonlinear transformation of the original unknown solution.

The transformation is derived by ignoring spatial dependence, which is appropriate if the solution is slowly varying. It gives an *equivalent* formulation of the problem and hence it does *not* restrict the type of blow-up that can occur. The blow-up is determined dynamically by solving the full reformulated problem. The transformation can be derived more generally, and this will be discussed elsewhere [11].

Under some reasonable assumptions, we determine the asymptotic form, time and position of the blow-up. The asymptotic form shows that in the reformulated problem, all the terms are bounded. The asymptotic results are compared with numerical simulations of the full reformulated equation.

Traditionally, the main computational tool has been a dynamical rescaling of time, space and solution that takes advantage of the scale invariant structure of the equations to keep the numerical solutions bounded (see [2,4,12]). In our method, since all the terms in the reformulated equation are bounded, we can use straightforward finite differences to solve the problem numerically. The numerical solutions are resolved right up to the singularity time. We find good agreement between the asymptotic and the numerical results for the structure, time, and position of the singularity.

In section 2 the asymptotic results are derived and in section 3 the numerical results are presented. Section 4 contains some conclusions.

## 2. Reformulation and Asymptotics

We consider the class of semilinear heat equations

$$u_t = \Delta u + f(u); \;\; u(x,0) = u_0(x) > 0; \;\; x \in R^n \tag{1}$$

where

$$\frac{f(u)}{u} \to \infty \text{ as } u \to \infty \tag{2}$$

Examples of this type of nonlinearity include $f(u) = u^p$ for $p > 1$ and $f(u) = e^u$, both of which occur in models of combustion [1]. We expect this class of problems to have blowing-up solutions where the blow-up is isolated and has a characteristic structure. For exact statements of theorems, see [1,5,6,9,10] for example.

Our interest is to study the asymptotic behavior of solutions which blow up at the time $t_c$ and at the point $x_c$. To do so we introduce a new unknown $v$ defined by

$$v(x,t) = \int_{u(x,t)}^{\infty} \frac{d\tilde{u}}{f(\tilde{u})} \tag{3}$$

Then we have the following *formal* asymptotic results:

**Asymptotic Results**

1. *Suppose that $v(x,t)$ is asymptotically spherically symmetric about $x_c$ and is an even function of $|x-x_c|$. Then $v$ has the following asymptotic form:*

$$v(x,t) \sim (t_c - t) - \frac{n}{2}\int_0^{t_c-t} |log f(u(s))|^{-1} ds + \frac{|x-x_c|^2}{4|log f(u(t_c-t))|} + \cdots \tag{4}$$

2. *Suppose that the initial function is slowly varying, i.e. that it is of the form $u_0(\epsilon x)$ with $\epsilon$ small, and that it attains its maximum at $x_m$. Let $T_0(x) = \int_{u_0(x)}^{\infty} \frac{d\tilde{u}}{f(\tilde{u})}$. Then*

$$t_c = T_0(x_m) + \epsilon^2 T_0(x_m)\Delta T_0(x_m) + O(\epsilon^4)$$

$$x_c = x_m - \epsilon^2 \frac{T_0(x_m)\nabla\Delta T_0(x_m)}{\Delta T_0(x_m)} + O(\epsilon^4) \tag{5}$$

## A. Derivation of Transformation

Omit the $\Delta u$ and let the nonlinearity be given by $u^p$ in equation (1). This reduces (1) to the ODE problem

$$u_t = u^p, \quad u(x,0) = u_0(x) > 0, \quad x \epsilon R^n. \tag{6}$$

This is exactly solvable with the solution:

$$u(x,t) = (p-1)^{\frac{1}{1-p}} (T_o(x) - t)^{\frac{1}{1-p}} . \tag{7}$$

Here $T_0(x)$ is defined by

$$T_0(x) = \frac{u_0(x)^{1-p}}{p-1} \tag{8}$$

Clearly $u$ blows up as $t \to \min T_0(x)$.

This example motivates the nonlinear transformation $u \to v$:

$$u(x,t) = [(p-1)v(x,t)]^{\frac{1}{1-p}} \tag{9}$$

Thus the blow-up of $u$ is replaced by the vanishing of $v$ in a special way. By (7), we expect $v$ to vanish linearly in time as $t \to t_c$, the blow-up time. Reformulating (1) in terms of $v$ gives:

$$v_t = -1 + \Delta v - \frac{p}{p-1}\frac{|\nabla v|^2}{v}\ , \ v(x,0) = (p-1)^{-1}[u_0(x)]^{1-p}. \tag{10}$$

Thus, the effect of the transformation is to replace the $u^p$ term in (1) by $-1$ (this is the linear vanishing) but at the expense of adding the nonlinear gradient term. This additional term might be cause for worry as $v$ appears in the denominator and is expected to vanish. However, we will show later that the asymptotic structure of $v$ near the blow-up point is such that this term is in fact bounded.

Now, suppose that we have the general nonlinearity $f(u)$. Similar considerations suggest the following nonlinear transformation:

$$v(x,t) = \int_{u(x,t)}^{\infty} \frac{d\tilde{u}}{f(\tilde{u})} \tag{11}$$

which gives the reformulated problem

$$v_t = -1 + \Delta v - f'(u(v))|\nabla v|^2\,,\; v(x,0) = \int_{u_0(x)}^{\infty} \frac{d\tilde{u}}{f(\tilde{u})}. \tag{12}$$

An advantage of the transformation is that the main effect of the nonlinearity is scaled out of the equation for $v$. The $v$ equation depends only weakly on the nonlinearity $f(u)$. The original solution $u$ is obtained by inverting the transformation (11), either analytically or numerically.

### B. Asymptotic Structure in Time

Let us suppose that the first singularity appears at $x = 0$ (without loss of generality) and at $t = t_c$. Let $v$ be given by (11) and satisfy equation (12). Further, *suppose* that near the singularity, $v$ has an even, radial expansion:

$$v(x,t) = v_0(t) + \frac{r^2}{2!} v_2(t) + \frac{r^4}{4!} v_4(t) + \cdots \tag{13}$$

where $r = |x|$. Now, substituting the expansion (13) of $v$ into equation (12) and equating coefficients of $r$, we obtain the coupled differential equations for the coefficients $v_0$ and $v_2$:

$$\begin{aligned} \frac{dv_0(t)}{dt} &= -1 + nv_2 \\ \frac{dv_2(t)}{dt} &= -2f'(u(v_0))v_2^2 + \frac{2+n}{3} v_4 \end{aligned} \tag{14}$$

Of course, the system is not yet closed as $v_4$ appears explicitly and thus $v_0$ and $v_2$ depend on the higher coefficients through $v_4$. However, let us *suppose* that $|v_4| << |f'(u(v_0))v_2^2|$ near the singularity time. Then omitting $v_4$ closes the system (14) near the singularity time as:

$$\begin{aligned} \frac{dv_0(t)}{dt} &= -1 + nv_2 \\ \frac{dv_2(t)}{dt} &= -2f'(u(v_0))v_2^2 \end{aligned} \tag{15}$$

This system can be solved in the following manner. First, we rewrite (15) as

$$\begin{aligned} v_0(t) &= (t_c - t) - n\int_t^{t_c} v_2(\tau)d\tau \\ v_2(t) &= \left[2\int_0^t f'(u(v_0(\tau)))d\tau + \frac{1}{v_2(0)}\right]^{-1} \end{aligned} \tag{16}$$

by integrating in time. We can solve (16) by the iteration method:

$$\begin{aligned} v_0^{j+1}(t) &= (t_c - t) - n\int_t^{t_c} \left[2\int_0^{\tau} f'(u(v_0^j(\tilde\tau)))d\tilde\tau + \frac{1}{v_2(0)}\right]^{-1} d\tau \\ v_2^{j+1}(t) &= \left[2\int_0^t f'(u(v_0^{j+1}(\tau)))d\tau + \frac{1}{v_2(0)}\right]^{-1} \end{aligned} \tag{17}$$

We use the starting values

$$\begin{aligned} v_0^0(t) &= t_c - t \\ v_2^0(t) &= \frac{1}{2}|\log f(u(t_c - t))| \end{aligned} \tag{18}$$

They are suggested by eliminating the $v_2$ term from the first equation in (16) to decouple the two equations. See the appendix for the evaluation of the integral in the second equation. That this method is convergent, for $t_c - t$ small enough, and asymptotic, in the sense that

$$\frac{v_0^{j+1}}{v_0^j} = 1 + G_{j,j+1}(t), \text{ where } G_{j,j+1}(t) \to 0 \text{ as } t \to t_c \tag{19}$$

will be presented elsewhere [11].

By using (19), the starting conditions (18), and the expansion (13), we find that:

$$\begin{aligned} v(x,t) &= \int_{u(x,t)}^{\infty} \frac{d\tilde u}{f(\tilde u)} \\ &\sim (t_c - t) - \frac{n}{2}\int_0^{t_c - t} |log f(u(s))|^{-1} ds + \frac{r^2}{4|log(f(u(t_c - t))|} + \cdots \end{aligned} \tag{20}$$

In the case $f(u) = u^p$, (20) reduces to:

$$v(x,t) \sim (t_c - t) - \frac{n}{2p}\int_0^{(p-1)(t_c-t)} \frac{dx}{|log x|} + \frac{p-1}{4p}\frac{r^2}{|log(t_c - t)|} + \cdots \tag{21}$$

In terms of $u$, (21) gives

$$u(x,t) \sim \left\{(p-1)\left[t_c - t - \frac{n}{2p}\int_0^{(p-1)(t_c-t)} \frac{dx}{|log x|} + \frac{p-1}{4p}\frac{r^2}{|log(t_c - t)|} + \cdots\right]\right\}^{\frac{1}{1-p}} \tag{22}$$

Alternatively, in the case $f(u) = e^u$, we get:

$$u(x,t) \sim -log\left[(t_c - t) - \frac{n}{2}\int_0^{t_c - t}\frac{dx}{|log x|} + \frac{r^2}{4|log(t_c - t)|} + \cdots\right], \qquad (23)$$

since the inverse of the transformation is $u(x,t) = -log(v(x,t))$. By expanding (22), using the binomial theorem, we see that it agrees with the results in [5,7-10]. Similarly (23) agrees with those derived in [3]. An advantage of our approach is that it yields the more general result (20) for $v$. Inverting the transformation (11), either analytically or numerically, to find $u$ determines its general asymptotic structure. This completes the derivation of (4).

### C. Slowly Varying Initial Data

In the preceding analysis we assumed that blow-up occurs at some point $t_c, x_c$, but we did not determine that point. By considering slowly varying initial data we shall show that blow-up does occur at one point, and we shall determine it.

We suppose that $u(x,0) = u_0(\epsilon x)$. Then letting $x' = \epsilon x$ in (1) and then omitting the prime gives

$$u_t = \epsilon^2 \Delta u + f(u) \qquad (24)$$

Defining $v$ by (11) as before, and transforming (24) correspondingly gives

$$v_t = -1 + \epsilon^2 \Delta v - \epsilon^2 f'(u(v))|\nabla v|^2 \qquad (25)$$

The initial data for $v$ are given by

$$v(x,0) = T_0(x) = \int_{u_0(x)}^{\infty} \frac{d\tilde{u}}{f(\tilde{u})} \qquad (26)$$

Thus (1) has been transformed to (25) and (26) for $v$.

We now suppose that $v$ has the asymptotic form

$$v(x,t,\epsilon) = v_0(x,t) + \epsilon^2 v_2(x,t) + O(\epsilon^4) \qquad (27)$$

Only even powers of $\epsilon$ appear since (25) depends on $\epsilon^2$. Substituting (27) into (25) and equating powers of $\epsilon$ gives the system

$$\begin{aligned} \frac{dv_0(t)}{dt} &= -1; \quad v_0(x,0) = T_0(x) \\ \frac{dv_2(t)}{dt} &= \Delta v_0 - f'(u(v_0))|\nabla v_0|^2; \quad v_2(x,0) = 0 \end{aligned} \qquad (28)$$

These equations (28) for the first two coefficients can be solved in closed form. Substituting the results into the expansion (27) gives

$$v(x,t) \;=\; T_0(x) - t + \epsilon^2 \left[ t\Delta T_0(x) - |\nabla T_0(x)|^2 log \frac{f(u(T_0(x)-t))}{f(u_0(x))} \right] + O(\epsilon^4) \tag{29}$$

We now seek the first time at which $v(x,t) = 0$ for some $x$. To do so we evaluate (29) at $x_c, t_c$ and equate it to zero:

$$\begin{aligned} v(x_c,t_c) = 0 \;&=\; T_0(x_c) - t_c \\ &\quad + \epsilon^2 \left[ t_c\Delta T_0(x_c) - |\nabla T_0(x_c)|^2 log \frac{f(u(T_0(x_c)-t_c))}{f(u_0(x_c))} \right] + O(\epsilon^4) \end{aligned} \tag{30}$$

Without loss of generality, we assume that $T_0(x)$ attains its minimum value at $x = 0$. We also assume that $\nabla T_0(0) = 0$. Now we write the following expansions for $t_c$ and $x_c$:

$$\begin{aligned} t_c &= T_0(0) + \epsilon^2 t_{c,2} + \cdots \\ x_c &= 0 + \epsilon^2 x_{c,2} + \cdots \end{aligned} \tag{31}$$

We use the expansions (31) in (30) and equate powers of $\epsilon$. At the second order, this yields only the correction to the blow-up time:

$$t_{c,2} = T_0(0)\Delta T_0(0) \tag{32}$$

The second order correction to the blow-up position is still undetermined. To find it, we take the gradient of (29) and suppose further that $(x_c, t_c)$ is a critical point of $v$. Then

$$\begin{aligned} \nabla v(x_c,t_c) \;&=\; 0 \\ &=\; \nabla T_0(x_c) + \epsilon^2 t_c \nabla \Delta T_0(x_c) \\ &\quad - \epsilon^2 \nabla \left\{ |\nabla T_0(x_c)|^2 log \frac{f(u(T_0(x_c)-t_c))}{f(u_0(x_c))} \right\} + O(\epsilon^4) \end{aligned} \tag{33}$$

Now, using the expansions (31) in (33) determines the second order correction to the position:

$$x_{c,2} = \frac{-T_0(0)\nabla\Delta T_0(0)}{\Delta T_0(0)} \tag{34}$$

This completes the derivation of (5).

### 3. Numerical Results

We now investigate, numerically, the behavior of solutions to (1) with $f(u) = u^5$ and in one space dimension. This problem was most recently studied in [2]. We compare the asymptotic results 4, 5 with the numerical simulations of the full reformulated problem. The problem we solve is:

$$\begin{aligned} u_t &= u_{xx} + u^5; \qquad -a \le x \le a \\ u(\pm a, t) &= u_0(\pm a) \end{aligned} \tag{35}$$

These boundary conditions, were chosen for their simplicity. We found that the boundary conditions had little effect on the singularity structure provided that the boundary is far enough away from the maximum of $u_0$.

In this case, $p = 5$ and the appropriate transformation is

$$u(x,t) = [4v(x,t)]^{-\frac{1}{4}} \tag{36}$$

The reformulated problem is then

$$\begin{aligned} v_t &= -1 + v_{xx} - \tfrac{5}{4} v_x^2/v; \qquad -a \le x \le a \\ v(\pm a, t) &= T_0(\pm a) = u_0(\pm a)^{-4}/4 \end{aligned} \tag{37}$$

We now use the asymptotic result (4) with $f(u) = u^5$. When the second term is omitted, because it is asymptotically smaller than the first term, we obtain

$$v \sim (t_c - t) + \frac{(x - x_c)^2}{5|log(t_c - t)|} \tag{38}$$

To estimate $t_c$ and $x_c$ we use (5) with $\epsilon = 1$ and we get

$$\begin{aligned} t_c &= T_0(x_m) + T_0(x_m) T_0''(x_m) \\ x_c &= x_m - \frac{T_0(x_m) T_0'''(x_m)}{T_0''(x_m)} \end{aligned} \tag{39}$$

The 'higher order' corrections have been omitted. We cannot expect 39 to be very accurate since $\epsilon = 1$ is not small.

It is straightforward to see that the asymptotic form of $v$ implies that the nonlinear term in 37 is bounded. Let

$$v_a(x,t) = (t_c - t) + \frac{x^2}{5|log(t_c - t)|} \tag{40}$$

Then

$$\frac{v_{ax}^2}{v_a} = \frac{4}{5} \frac{x^2}{[x^2 + 5(t_c - t)|log(t_c - t)|]} \tag{41}$$

This is clearly bounded, even as $t \to t_c$. Therefore, we shall use straightforward finite differences to solve the problem numerically.

We define $w_j^n$ to be the numerical approximation to $v(n\Delta t, jh)$ where $\Delta t$ and $h$ are the temporal and spatial grid sizes respectively. The difference equation is

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = -1 + \frac{(w_{j-1}^n - 2w_j^n + w_{j+1}^n)}{h^2} - \frac{5}{16}\frac{(w_{j+1}^n - w_{j-1}^n)^2}{h^2}\frac{1}{w_j^n} \tag{42}$$

which is first order in time and second order in space. We also choose $\Delta t = \frac{1}{4}h^2$ to satisfy the stability requirement.

The numerical blow-up time $t_c^*$ is chosen to be the first time that the numerical solution $v$ becomes negative, and $x_c^*$ is the position at which this occurs. A typical value of the numerical solution at $t_c^*$ is $-10^{-8}$, so $t_c^*$ is only a slight overestimate. Thus, the calculation actually continues the solution slightly beyond the blow-up time of the original solution $u$. The method becomes unstable a short time after the blow-up happens, however. The numerical blow-up structure will be compared to the predicted formula 38 of the parameters $t_c$ and $x_c$.

The first calculation is the evolution of the parabola

$$v(x,o) = T_0(x) = 2^{-6} + \frac{x^2}{2}; \quad -1.5 \le x \le 1.5 \tag{43}$$

Here $x_m = 0$, $T_0''(0) = 1$ and $T_0'''(0) = 0$. The time evolution of the solution $v(x,t)$, with $h = .00586$, is shown in figure 1. The topmost curve is the initial condition ($t = 0$) and the lower curves are the solutions at successively later times. At the final time shown, the minimum value of $v$ is $O(10^{-8})$. Note that the profile tends to flatten. This corresponds to a sharpening profile of the original untransformed solution $u$ as is shown in figure 2. For figure 2, $v$ was computed first and then $u$ was determined from $v$ via the transformation 36. Notice how sharply peaked the final profile is in figure 2. The final curve in figure 2 is shown at a slightly later time than that of figure 1. In figure 3, the values of $u$ on the grid at the final time level are shown. It is clear from their distribution that a direct computation of $u$ is very difficult as numerical resolution and stability become increasingly difficult to maintain. This is in contrast to figure 4 which shows the distribution of $v$ on the grid. There is clearly no such problem here. This is the remarkable feature of the reformulation!

We now compare the asymptotic structure of the singularity (given by 38) with the numerical computation in figure 1. The result is shown in figure 5. The agreement is excellent and the curves lie on top of each other (at the singular point) to within the plotting resolution. The asymptotic predictions 39 for these data are $t_c = 3.125 \times 10^{-2}$ and $x_c = 0$. The computed values are found to be $t_c^* = 2.861 \times 10^{-2}$ and $x_c^* = 0$. The agreement for $t_c$ is poor because $\epsilon = 1$ is not small.

The previous computation, as well as all the subsequent computations, takes less than 5 minutes to run on a Sun sparcstation 2. In the subsequent computations, we do not give the graphs of the original solution $u$ as their form is the obvious modification of that from the parabolic computation.

We now consider initial data with more than one local minimum (maximum for the original solution $u$). Such data presents a challenge for rescaling algorithms as it is not always clear which maximum will blow-up first.

The second computation has as initial data the fifth order polynomial satisfying the conditions

$$\begin{aligned} T_0(\pm.5) &= .015625 = 2^{-6} \\ T_0'(\pm.5) &= 0 \\ T_0''(\pm.5) &= 1 \end{aligned} \tag{44}$$

which has local minima at $x = \pm.5$. The domain is $1.5 \leq x \leq 1.5$ and the third derivative is found to be $T_0'''(\pm.5) = \pm 6$. The time evolution of this solution is shown in figure 6 ($h = .00586$). Both minima tend to flatten and again there is no problem with resolution or stability. The minima pass through zero at the same time.

The asymptotic structure 38 is compared with the numerical results in figure 7 and again the agreement is excellent. The asymptotic results 39 predict that both minima will hit zero at the same time $t_c = 3.125 \times 10^{-2}$ and their positions will be $x_c = \pm.40625$. Numerically, both minima pass through zero at $t_c^* = 2.6485 \times 10^{-2}$ and at $x_c^* = \pm.428175$. Of course since $\epsilon = 1$ is not small, the lack of agreement is to be expected.

The third computation takes initial data with two local minima, but of different heights:

$$\begin{aligned} T_0(-.5) &= .015625 \\ T_0(+.5) &= .02 \\ T_0'(\pm.5) &= 0 \\ T_0''(\pm.5) &= 1.0 \end{aligned} \tag{45}$$

on the domain $-1.5 \leq x \leq 1.5$. The third derivatives are $T_0'''(-.5) = -5.7375$ and $T_0'''(+.5) = 6.2625$. The time evolution of this solution is shown in figure 8 ($h = .00293$). Both minima still tend to flatten, but the one corresponding to $x = -.5$, initially, passes through zero first.

The asymptotic results predict that the minimum located at $x = -.5$, initially, will hit zero first at $t_c = 3.125 \times 10^{-2}$ and at $x_c = .41305$. The numerical singularity time and position are found to be $t_c^* = 2.663 \times 10^{-2}$

and $x_c^* = -.4248$. The asymptotic structure 38 is compared to the numerical result in figure 9. The agreement is excellent.

The fourth computation takes initial data with two local minima with the same height, but different curvatures

$$\begin{aligned} T_0(\pm.5) &= .015625 \\ T_0'(\pm.5) &= 0 \\ T_0''(+.5) &= 1.0 \\ T_0''(-.5) &= 1.5 \end{aligned} \tag{46}$$

on the domain $-1.5 \leq x \leq 1.5$. Further, $T_0'''(-.5) = -10.5$ and $T_0'''(.5) = 4.5$. For this data, the asymptotic prediction is that the minimum located at $x = .5$ initially, will hit zero first at $t_c = 3.125 \times 10^{-2}$ and at $x_c = .440625$. This is indeed what happens numerically. The evolution is shown in figure 10. The numerical singularity time and position are found to be $t_c^* = 2.6901 \times 10^{-2}$ and $x_c^* = .43945$. The asymptotic structure 38 is compared to the numerical result in figure 11 where the agreement is excellent.

It is not surprising that the predicted singularity times and positions do not agree so well with their numerical counterparts. The asymptotic predictions were derived only for slowly varying initial data, an assumption t38s clearly violated by the initial data for the numerical computations. It is to be expected, however, that the predicted asymptotic form (38a) agrees so well with the full numerical simulations. The asymptotic form was derived under the assumption that $t_c - t$ is small. The singularity time, $t_c$, and position, $x_c$, are assumed to be given. In the comparison, the numerical values $t_c^*$ and $x_c^*$ are used to determine the asymptotic form. Since this form is then compared to the numerical solution at times near $t_c^*$, we have every reason to expect the agreement to be good.

## 4. Conclusions

We have presented asymptotic and numerical results for blowing-up solutions to a class of semilinear heat equations. We have shown how to reformulate the problem exactly to yield a new equation. This new equation has, as its essential feature, all of its terms bounded; a fact that can be seen from the asymptotics. Thus, straightforward finite difference techniques can be used to solve it numerically. This is a tremendous savings as previous numerical computations, for this type of problem, have relied on dynamical rescaling algorithms [2,4,12] and are therefore much more complicated. In future work, we will consider more general transformations for this problem. We hope that by deriving the transformation more generally, we will be able to apply our techniques to other problems with blowing-up solutions. Of course, as we have seen here, the success of such a transformation depends crucially on the asymptotic form of the blow-up up solution. Finally, we hope that the reformulation will also help to simplify the proofs of rigorous results. This is partially borne out in [11] where we give a simple proof of blow-up.

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## 5. Appendix

In this appendix, we evaluate the integral

$$\int_0^t f'(u(t_c - \tau))d\tau \tag{47}$$

as follows. Let $v_0 = t_c - \tau$. Then $dv_0 = -d\tau$. Therefore the integral becomes

$$-\int_{v_0(0)}^{v_0(t)} f'(u(v_0))dv_0 \tag{48}$$

Now, recall that $u$ and $v_0$ are related by the transformation

$$v_0 = \int_u^\infty \frac{d\tilde{u}}{f(\tilde{u})} \tag{49}$$

Change the variable of integration in (47) to $\tilde{u} = u(v_0)$. Thus $d\tilde{u} = -f(\tilde{u})dv_0$. This gives

$$\int_{u(v_0(0))}^{u(v_0(t))} \frac{f'(\tilde{u})}{f(\tilde{u})}d\tilde{u} = log\frac{f(u(t_c - t))}{f(u_0)} \tag{50}$$

since $u(v_0(0)) = u_0$.

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## 6. Figure Captions

**Figure 1.** The time evolution of the parabolic initial data 43 is shown. The full computational domain is $-1.5 \le x \le 1.5$. ($h = .00586$, $t_c^* = 2.861 \times 10^{-2}$ and $x_c^* = 0$)

**Figure 2.** The time evolution of the original solution $u$ (found by transforming $v$, of figure 1, by 36. ($h = .00586$)

**Figure 3.** The distribution of $u$ on the grid, from figure 2, is shown.

**Figure 4.** The distribution of $v$ on the grid is shown.

**Figure 5.** A comparison of the asymptotic form 38 with the full numerical simulation (from figure 1 at $t = 2.8607 \times 10^{-2}$) is shown. The parameters $t_c$ and $x_c$, in formula 38 are replaced by $t_c^*$ and $x_c^*$ respectively.

**Figure 6.** The time evolution of the solution with data 44 is shown. The full computational domain is $-1.5 \le x \le 1.5$. ($h = .00586$, $t_c^* = 2.6485 \times 10^{-2}$ and $x_c^* = \pm.428175$)

**Figure 7.** A comparison of the asymptotic form with the full numerical simulation (from figure 6 at $t = 2.648 \times 10^{-2}$) is shown.

**Figure 8.** The time evolution of the solution with data 45 is shown. The full computational domain is $-1.5 \le x \le 1.5$. ($h = .00293$, $t_c^* = 2.6633 \times 10^{-2}$ and $x_c^* = -.4248$)

**Figure 9.** A comparison of the asymptotic form with the full numerical solution (from figure 8 at $t = 2.6631 \times 10^{-2}$) is shown.

**Figure 10.** The time evolution of the solution with data 46 is shown. The full computational domain is $-1.5 \leq x \leq 1.5$. ($h = .00293, t_c^* = 2.6901 \times 10^{-2}$ and $x_c^* = .43945$)

**Figure 11.** A comparison of the asymptotic form with the full numerical solution (from figure 10 at $t = 2.69 \times 10^{-2}$) is shown.