Surface phase separation and flow in a simple model of drops and vesicles

Tutorial Lecture 4

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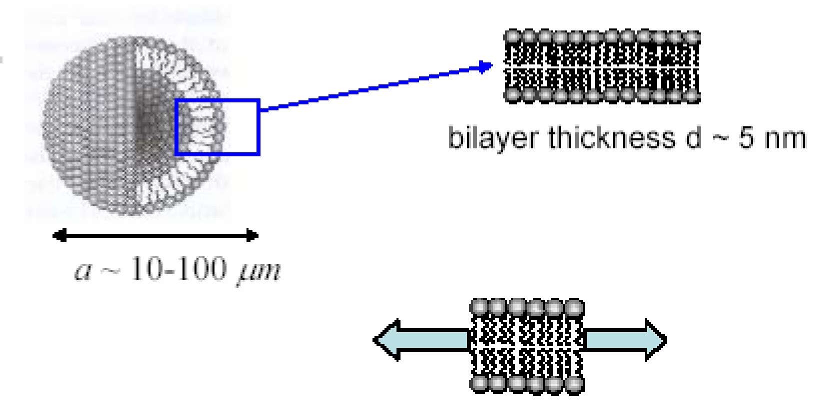
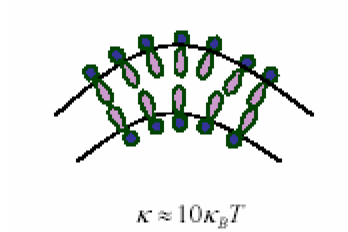
Biomembranes

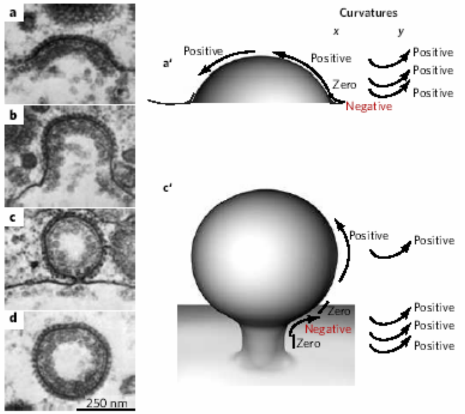




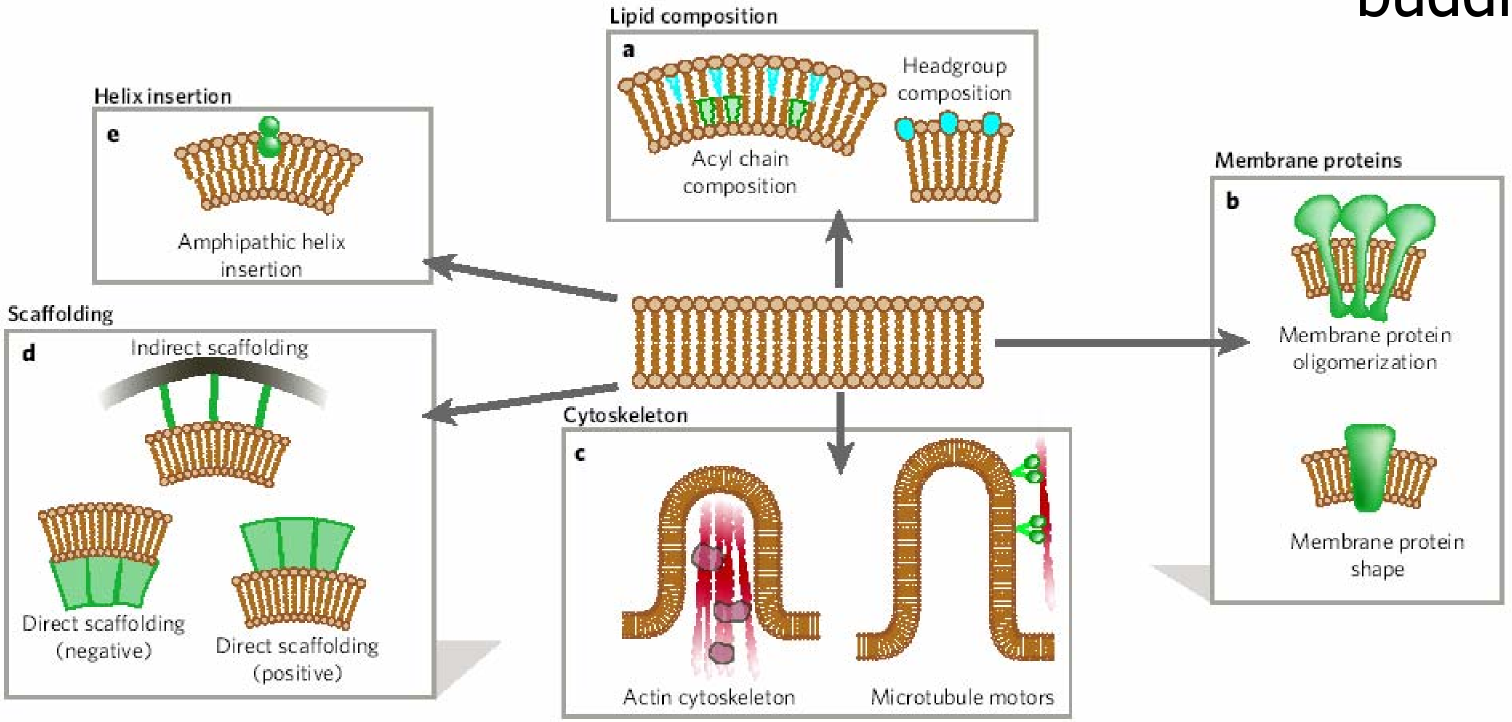
- Complex structures containing lipids, proteins, cholesterol, ions, etc
- Cell-boundary. Carrier vesicles. Several *nm* thick. Surface area can be m*m*.
- Active role in locomotion, adhesion solute/chemical transport, signal transduction, etc.

Structure ⟷ Morphology ⟷ Biological function

Mechanisms of conformation

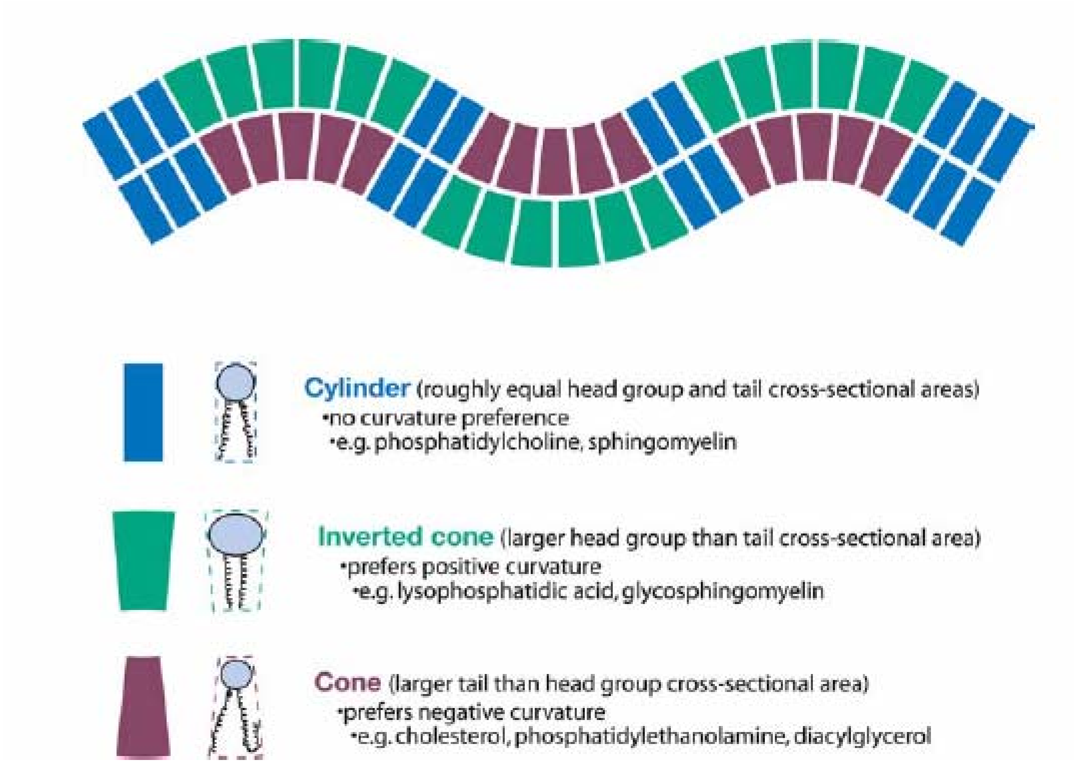


budding



McMahon, Gallop. Nature (2005)

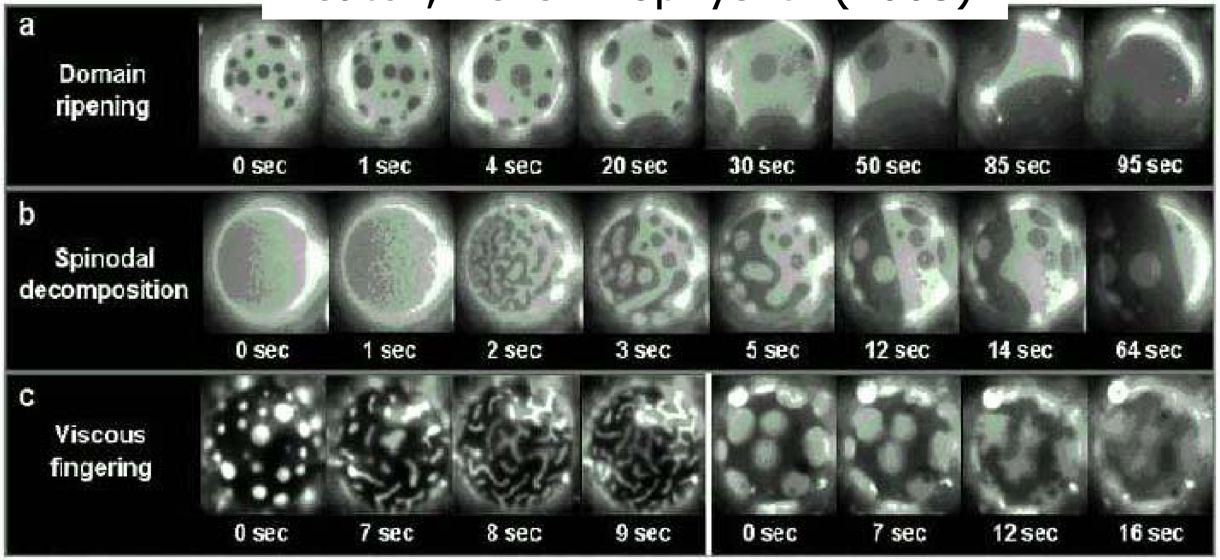
Lipid shape affects curvature



Mukherjee and Maxfield, Ann. Rev. Cell Dev. Biol. (2004)

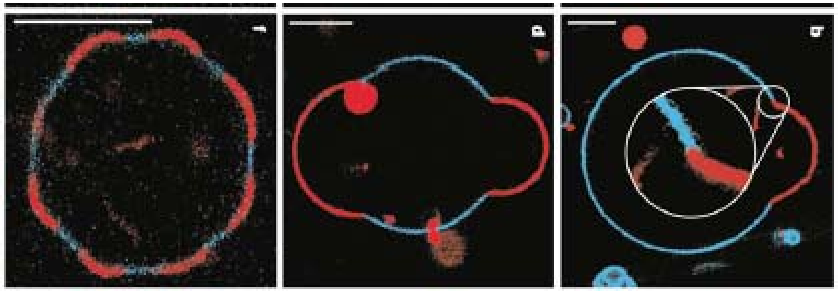
Multicomponent membranes

Veatch, Keller Biophys. J. (2003).



- Multiple lipid components
- Phase-separation/ domain formation
- Spinodal decomposition

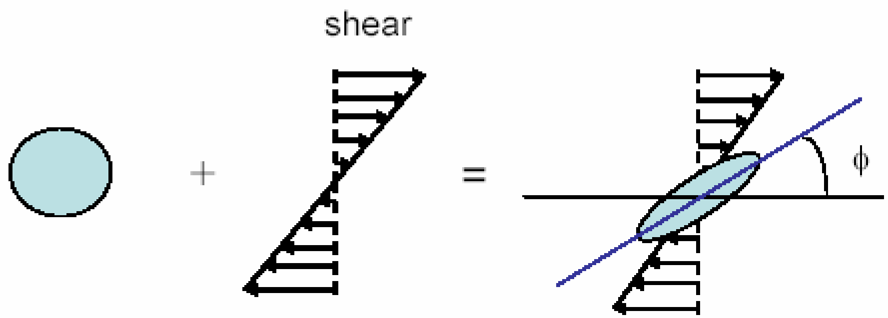
- Morphology (e.g. curvature) nonlinear coupled to surface composition of phases



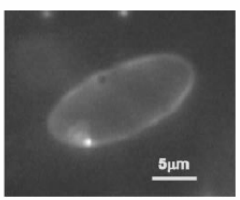
Baumgart et al, Nature (2003).

Effect of fluid flow

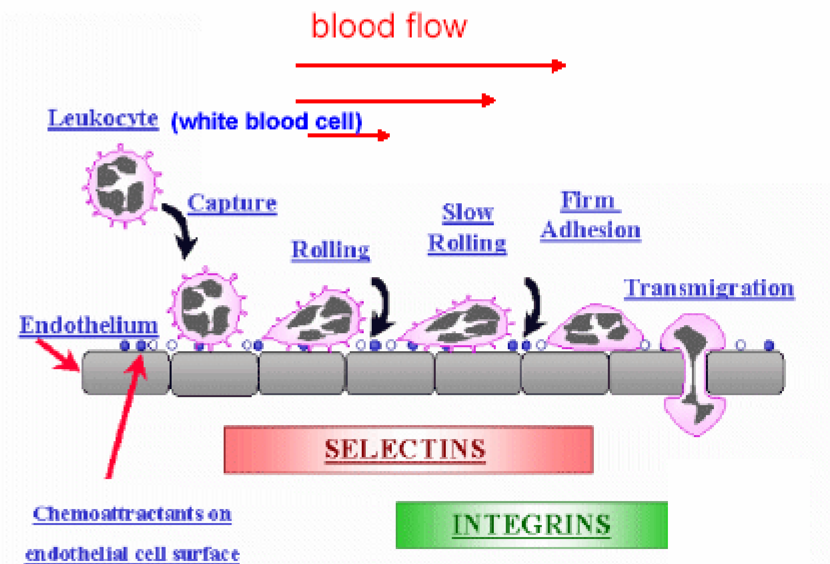
- Membranes/vesicles contain and are immersed in viscous liquid



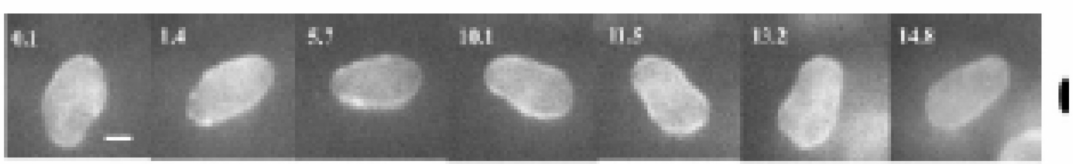
Low shear: Tank-treading



Kantsler and Steinberg, PRL **95** (2005)



High shear: Tumbling



Kantsler and Steinberg, PRL **96** (2006)

- Effect on spinodal decomposition/ phase-separation of multicomponent membranes??

Mathematical model

Generalized Helfrich model

(Zhong-can, Helfrich, Phys. Rev A 1989)

Membrane energy: $E_M = E_T + E_B + E_G + E_S$

Line tension: $E_T = \int_{\Sigma(t)} \left(g(f) + \frac{\epsilon^2}{2} |\nabla_s f|^2 \right) \, d\Sigma$

g: Double-well potential

Surface energy: $E_s = \int_{\Sigma(t)} \gamma(f) \, d\Sigma$

Bending energy: $E_b = \frac{1}{2} \int_{\Sigma} b_n(f) \left(\kappa - \kappa_0(f) \right)^2 d\Sigma$

κ mean curvature

Gaussian bending energy: $E_G = \int_{\Sigma} b_G(f) \, G \, d\Sigma$

G Gaussian curvature

f =mass concentration of component

Conservation of surface component

$$M(t) = \int_{\Sigma(t)} f \; d\Sigma = M(0)$$

Local conservation (Eulerian):

$$f_t + \mathbf{u} \cdot \nabla f - \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} f = \nabla_s \cdot \mathbf{J}_s \qquad \nabla_s = (\mathbf{I} - \mathbf{nn})\nabla$$

advection stretching generalized diffusion

$\mathbf{u}$: fluid velocity

Note: $-\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} = \nabla_s \cdot \mathbf{u}_s + \kappa \mathbf{u} \cdot \mathbf{n} \qquad \mathbf{u}_s = (\mathbf{I} - \mathbf{nn})\mathbf{u}$

- Determine constitutive relation for diffusion flux $\mathbf{J}_s$

Fluid flow: Stokes (low Reynolds number)

$$\nabla \bullet \mathbf{T}_i = 0,\ \nabla \bullet \mathbf{u}_i = 0 \qquad i=d,\ m$$

Stess tensor $\mathbf{T}_i = -p_i \mathbf{I} + \eta_i \left(\nabla \mathbf{u}_i + \nabla \mathbf{u}_i^T \right)$

pressure viscosity

Interface boundary conditions:

$$0 = [\mathbf{u}]_\Sigma \equiv (\mathbf{u}|_{\Sigma,m} - \mathbf{u}|_{\Sigma,d})$$

$$[\mathbf{Tn}]_\Sigma = T_n \mathbf{n} + T_\mathbf{s},$$

Far-field bc: $\mathbf{u} = \mathbf{u}_\infty \quad \text{on} \quad \partial\Omega.$

- Determine constitutive relation for forces: T_n and $T_\mathbf{s}$

Energy variation gives constitutive conditions

(thermodynamic consistency)

Take time derivative. Equivalent to variation (f and S varied independently).

Surface energy

$$\dot{E}_S = \int_\Sigma \gamma' \dot{f} - \gamma \mathbf{n} \bullet \nabla \mathbf{u} \bullet \mathbf{n} \, d\Sigma$$

$$= \int_\Sigma \dot{f} \frac{\delta E_S}{\delta f} + \mathbf{u} \bullet \frac{\delta E_S}{\delta \boldsymbol{\Sigma}} \, d\Sigma$$

where

$$\frac{\delta E_S}{\delta f} = \gamma'$$

$$\frac{\delta E_S}{\delta \boldsymbol{\Sigma}} = \kappa \gamma \mathbf{n} - \nabla_s \gamma$$

Line tension

$$\dot{E}_T = \int_\Sigma \left(g'(f) - \varepsilon^2 \Delta_s f \right) \dot{f} - \varepsilon^2 \nabla_s f \bullet \nabla \mathbf{u} \bullet \nabla_s f - \left(g + \frac{\varepsilon^2}{2} |\nabla_s f|^2 \right) \mathbf{n} \bullet \nabla \mathbf{u} \bullet \mathbf{n} \, d\Sigma$$

$$= \int_\Sigma \dot{f} \frac{\delta E_T}{\delta f} + \mathbf{u} \bullet \frac{\delta E_T}{\delta \boldsymbol{\Sigma}} \, d\Sigma$$

Constitutive equations

Flux: $J = \nu \nabla_s \mu, \;\; \mu = \frac{\delta E}{\delta f}$

$$\text{where} \frac{\delta E}{\delta f} = g'(f) - \varepsilon^2 \Delta_s f + \gamma' + \frac{b_n^{'}}{2}\left(\kappa - \kappa_0\right)^2 - b_n\left(\kappa - \kappa_0\right)\kappa_0^{'}$$

$\frac{\delta E_b}{\delta f}$

Tangential force: $T_{\mathbf{s}} = -\left(\nabla_s \sigma + \nabla_s f \frac{\delta E_b}{\delta f}\right)$

Normal force: $T_n = \sigma + \frac{\delta E_b}{\delta \Sigma_n}$

s is a generalized surface tension

(omitted Gaussian bending, is easily included)

Thermodynamic consistency

Surface tension: (2D for simplicity)

$$\sigma = \gamma + g - \frac{\varepsilon^2}{2} |\nabla_s f|^2 - f \frac{\delta E}{\delta f}$$

Bending: (2D for simplicity. Hausser will give 3D.)

$$\frac{\delta E_b}{\delta \Sigma_n} = -\left(\partial_{ss} \left(b_n \left(\kappa - \kappa_0 \right) \right) + \frac{b_n}{2} \kappa \left(\kappa^2 - \kappa_0^2 \right) \right)$$

Energy dissipation:

$$\begin{aligned} \dot{E} \ &= \ -\int_{\Sigma(t)} \nu |\nabla_s \mu|^2 \, d\Sigma - \frac{1}{2} \int_{\Omega_d} \eta_d (\nabla \mathbf{u}_d + \nabla \mathbf{u}_d^T) : (\nabla \mathbf{u}_d + \nabla \mathbf{u}_d^T) \, d\mathbf{x} \\ &\quad - \frac{1}{2} \int_{\Omega_m} \eta_m (\nabla \mathbf{u}_m + \nabla \mathbf{u}_m^T) : (\nabla \mathbf{u}_m + \nabla \mathbf{u}_m^T) \, d\mathbf{x}, \end{aligned}$$

Remarks

- Flow, morphology and phase-decomposition are intimately coupled
- Phase-transformation on moving interface
  e.g., 4^{th} order nonlinear equation on a moving surface
- Highly challenging theoretically and numerically

Focus on a simple case

No bending forces: $b_n = b_G = 0$

Surface energy depends on f

Nondimensionalization:

length scale = drop radius a time scale = $a/\bar{U}$

$\tilde{\mathbf{x}} = \mathbf{x}/a,\ \tilde{\mathbf{T}}_i = \mathbf{T}_i/\bar{p},\ \tilde{\mathbf{u}}_i = \mathbf{u}_i/\bar{U},\ \tilde{\sigma} = \sigma/\bar{\sigma}$

$\bar{p} = \eta_m \bar{U}/a$ characteristic stress scale

$\bar{\sigma}$ characteristic surface tension scale

$\bar{\mu}$ characteristic chemical potential scale

Nondimensional parameters:

Capillary number: $Ca = \eta_m \bar{U}/\bar{\sigma}$

Cahn number: $\mathcal{C} = \epsilon/(a\sqrt{\bar{\mu}})$

Mach number: $\mathcal{M} = \bar{\sigma}/\bar{\mu}$

Peclet number: $Pe = \bar{\nu}\bar{\mu}/(a\bar{U})$

Nondimensional system

Stokes:

$$\tilde{\nabla} \cdot \tilde{\mathbf{T}}_i = 0, \qquad \tilde{\nabla} \cdot \tilde{\mathbf{u}}_i = 0$$

$$\tilde{\mathbf{T}}_i = -\tilde{p}_i \mathbf{I} + \lambda_i \left(\tilde{\nabla} \tilde{\mathbf{u}}_i + \tilde{\nabla} \tilde{\mathbf{u}}_i^T \right)$$

$\lambda_m = 1$ and $\lambda_d = \eta_d / \eta_m$

Interface BC:

$$[\tilde{\mathbf{T}} \tilde{\mathbf{n}}]_{\tilde{\Sigma}} = \frac{1}{Ca} \left(\tilde{\sigma} \tilde{\kappa} \tilde{\mathbf{n}} - \tilde{\nabla}_s \tilde{\sigma} \right)$$

(+ continuity)

surface tension:

$$\tilde{\sigma} = \frac{1}{\mathcal{M}} \left(\tilde{g}(f) - f \tilde{g}'(f) - \frac{\mathcal{C}^2}{2} |\tilde{\nabla}_s f|^2 + \mathcal{C}^2 f \tilde{\Delta}_s f \right) + \tilde{\tau}(f)$$

Surface phase:

$$f_t + \tilde{\mathbf{u}} \cdot \tilde{\nabla} f - \tilde{\mathbf{n}} \cdot \tilde{\nabla} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}} f = \frac{1}{Pe} \tilde{\nabla}_s \left(\tilde{\nu} \tilde{\nabla}_s \tilde{\mu} \right)$$

Chemical potential:

$$\tilde{\mu} = \tilde{g}'(f) - \mathcal{C}^2 \tilde{\Delta}_s f + \mathcal{M} \tilde{\gamma}'(f)$$

Interface:

$$\frac{d\tilde{\mathbf{x}}}{dt} \cdot \mathbf{n} = \tilde{\mathbf{u}}$$

Numerical approach

- Level-set method to capture interface
- Immersed interface method to solve the Stokes equations
- Non-stiff surface phase-field solver

Xu, Li, Lowengrub, Zhao. JCP (2006)

Lowengrub, Xu, Voigt FDMP in review.

Numerical method

Popular numerical methods for surface-tension mediated interfacial flows

--front-tracking/boundary integral method (e.g. *Hou, Lowengrub and Shelley, JCP, 2001*)

-- front-tracking/continuum surface force(CSF) method (e.g. *Glimm etal, JCP, 2001* , *Tryggvason etal, JCP, 2001*)

-- volume-of-fluid/CSF method (e.g. *Scardovelli and Zaleski, Ann. Rev. Fluid Mech., 1999*)

-- level-set/CSF method (e.g. *Osher and Fedkiw, JCP, 2001; Zheng, Lowenegrub, Anderson, Cristini* , 2005)

-- phase field method (e.g. *Anderson, McFadden and Wheeler, Ann. Rev. Fluid. Mech., 1988*)

-- other hybrid methods, such as volume-of-fluid/level-set method, particle level-set method

As opposed to the CSF methods, sharp interface flow solvers dealing with interface jump conditions without smoothing

-- immersed interface method (IIM) (e.g. *LeVeque and Li, SIAM J. Sci. Comp., 1997*)

-- ghost fluid method (e.g. *Fedkiw et al, JCP, 1999*)

-- others (*Mayo, Helenbrook, Martinelli and Law, JCP, 1999*)

Numerical method continued

Advantages of the level set method (*Osher and Sethian, JCP, 1988*)

-- Accurate representation of interface geometry

-- capable of handling topological change

-- relatively easy 3D implementation

Advantages of the IIM

-- no introduction of intermediate non-physical state near the interface

-- higher order accuracy as opposed to CSF method

-- fast Poisson solvers (e.g. FFT, multigrid) available for the discrete system

Numerical method continued

Relatively little work on surfactants. None on surface phase-decomposition on moving interface with flow

-- front-tracking/boundary integral method (e.g. *Milliken,Stone and Leal, Phys. Fluids, 1993*)

-- volume-of-fluid/CSF method (e.g. *Drumwrite-Clark and Renardy, Phys. Fluids, ; 2004, James and Lowengrub, JCP, 2005; Bothe and Alke 2006*)

-- front-tracking/CSF method (e.g. *Ceniceros, Phys. Fluids, 2003*)

-- level-set/immersed interface method LS/IIM
(*Xu, Li, Lowengrub and Zhao, JCP, 2006*).
Feature: a stable (large time step $\Delta t = O(h)$ and second-order accurate surfactant solver (*Xu and Zhao, J. Sci. Comp. 2003*) coupled with the second-order accurate IIM for flow solver in conjunction with the level-set method.

-- phase-decomposition on fixed, complex interfaces (Greer et al (2005), Voigt et al (2005))

-- phase-decomposition on moving interfaces (no flow). Wang and Du (2006); Ratz and Voigt (2006).

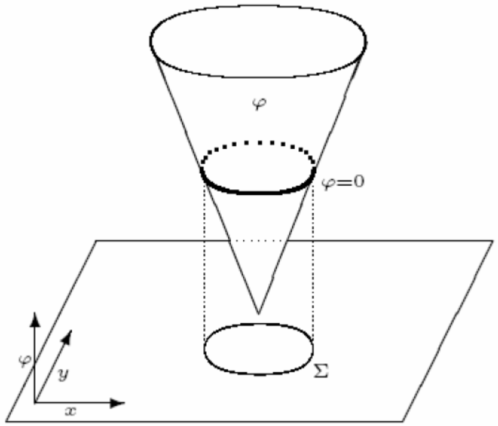
-- homogeneous membranes in flow: Siefert et al, Biben and Misbah (2003), Noguchi and Gompper (2005), Du et al. (2006)

- Extend LS/IIM algorithm for surfactant to case with surface phase-decomposition

Interface representation using a level set function

Convection of the level set function:

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$



The interface $\Sigma = \{\mathbf{x} : \phi(\mathbf{x}, t) = 0\}$

Assume $\phi(\mathbf{x}, t) < 0$ inside drops,

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}, \qquad \kappa = \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right)$$

Reinitialization of level set function:

$$\phi_\tau + S(\phi_0)(|\nabla \phi| - 1) = 0, \;\; \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x})$$

$$S(\phi) = \phi / |\phi| \quad \text{is the sgn function}$$

Reformulation of Stokes equations

Three Poisson equation approach:

Pressure Poisson equation

$$\nabla^2 p = 0, \qquad \frac{\partial p}{\partial \mathbf{n}} = \nabla^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on } \partial\Omega$$

$$[p]_\Sigma = -\frac{1}{Ca}\sigma\kappa, \qquad \left[\frac{\partial p}{\partial \mathbf{n}}\right]_\Sigma = \frac{1}{Ca}\nabla_s{}^2\sigma.$$

Velocity Poisson equations

$$\nabla^2 \mathbf{u} = \nabla p, \qquad \mathbf{u} = y\mathbf{e}_y \text{ on } \partial\Omega,$$

$$[\mathbf{u}]_\Sigma = 0, \qquad \left[\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right]_\Sigma = \frac{1}{Ca}\nabla_s \sigma.$$

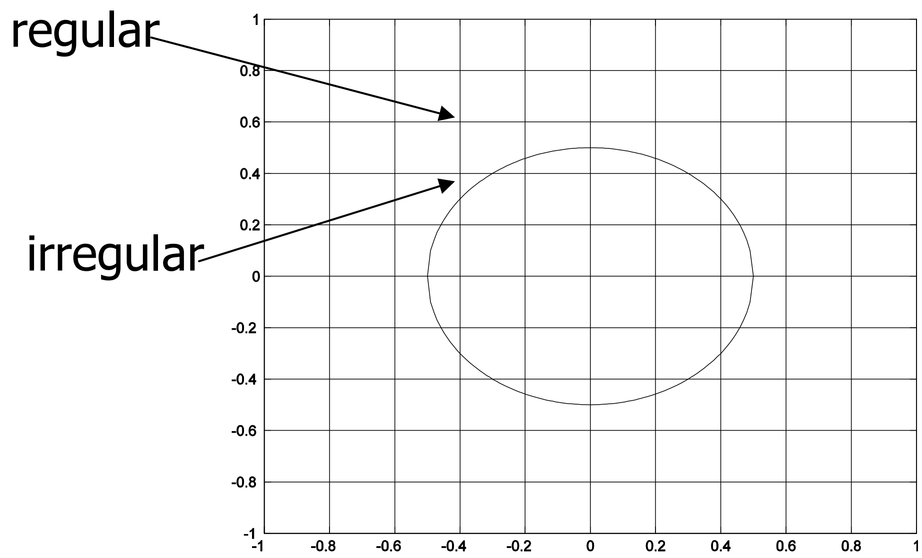
Numerical method continued: a brief introduction of IIM

The IIM (*LeVeque and Li, SIAM J. Numer. Anal., 1994)* for Poisson equation

$$\Delta w = f$$

with jump conditions

$$[w]_\Sigma,\ [\frac{\partial w}{\partial \mathbf{n}}]_\Sigma \quad \text{given}$$



All grid points divided into two groups: *regular* and *irregular*

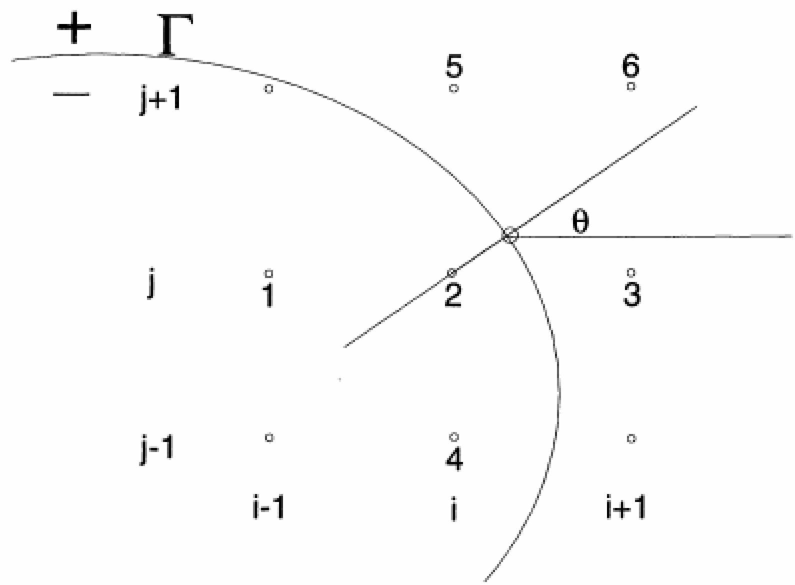
-- standard centered difference scheme at *regular* points.

-- modified centered difference scheme at *irregular* points by adding a correction term which involves jump conditions at the interface.

$$\sum_k \gamma_{ijk} w_{i+i_k, j+j_k} = f_{ij} + C_{ij}$$

Second order accuracy in maximum norm achieved.

IIM



- Select a point $(x_i^*, y_j^*) \in \Gamma$ near (x_i, y_j).
- Apply a local coordinate transformation in directions normal and tangential to Γ at (x_i^*, y_j^*).
- Derive the jump conditions relating $+$ and $-$ values at (x_i^*, y_j^*) in the local coordinates.
- Choose an additional point to form a six–point stencil.
- Set up and solve a linear system of six equations for the coefficients γ_k. The value C_{ij} is also obtained.

$$
\begin{aligned}
u(x_i, y_j) &= u^- + u_x^- (x_i - x_i^*) + u_y^- (y_j - y_j^*) + \tfrac{1}{2} u_{xx}^- (x_i - x_i^*)^2 \\
&\quad + \tfrac{1}{2} u_{yy}^- (y_j - y_j^*)^2 + u_{xy}^- (x_i - x_i^*)(y_j - y_j^*) + O(h^3)
\end{aligned}
$$

Numerical method continued: IIM for the Stokes equations

IIM for the pressure Poisson equation

$$\nabla^2 p^{k+1} = 0,$$

$$[p^{k+1}]_{\Sigma_{k+1}} = -\frac{1}{Ca}\sigma(f^{k+1})\kappa^{k+1}, \quad \left[\frac{\partial p^{k+1}}{\partial n}\right]_{\Sigma_{k+1}} = \frac{1}{Ca}\nabla_s^2\sigma(f^{k+1})$$

$$\left(\frac{\partial p}{\partial n}\right)^{k+1} = \frac{3}{2}\left(\nabla^2\mathbf{u}\cdot\mathbf{n}\right)^k - \frac{1}{2}\left(\nabla^2\mathbf{u}\cdot\mathbf{n}\right)^{k-1} \quad \text{on} \quad \partial\Omega$$

IIM for the velocity Poisson equations

$$\nabla^2\mathbf{u}^{k+1} = \nabla p^{k+1},$$

$$[\mathbf{u}^{k+1}]_{\Sigma^k} = 0, \quad \left[\frac{\partial\mathbf{u}}{\partial n}\right]_{\Sigma^k} = \left(\frac{1}{Ca}\nabla_s\sigma\right)^{k+1},$$

$$\mathbf{u}^{k+1} = y\mathbf{e}_y \quad \text{on} \quad \partial\Omega$$

Discretization of the interface jump conditions: with the level set extension methodology, κ, $\nabla_s\sigma$ and $\nabla_s^2\sigma$ are calculated at grid points, then interpolated at the interface

Numerical method continued: evolution of the level sets: Advection and reinitialization

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0.$$

-- high order WENO scheme (e.g. *Jiang and Peng, SIAM J. Sci. Comp. 2000*) for spatial discretization

-- high order TVD Runge-Kutta method (e.g. *Shu, SIAM J. Sci. Comp., 1988*) for time marching

-- smooth sign function used in the reinitialization

$$\begin{cases} \phi_\tau + S(\phi_0)(|\nabla \phi| - 1) = 0 \\ \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \end{cases}$$

$$\tilde{S}(\phi) = \frac{\phi}{\sqrt{\phi^2 + h^2}}$$

Remark: it is necessary to use high order schemes in order to accurately compute the normal and curvature of the interface

Numerical method continued: evolution of surfactant concentration

- Extension of surface phase off the interface

$$\begin{cases} f_\tau + S(\phi)\mathbf{n}\cdot\nabla f = 0, \\ f(\mathbf{x},0) = f_0(\mathbf{x}) \end{cases}$$

- A semi-implicit backward Euler method for surface phase equation to remove the stiffness

$$\Delta_s \mu = \Delta\mu - \frac{\partial^2 \mu}{\partial n^2} - \kappa\frac{\partial \mu}{\partial n};$$

implicit explicit

$$g'(f) = \bar{a} f + (g'(f) - \bar{a} f)$$

$$\frac{f^{n+1} - f^{n}}{\Delta t} - \frac{1}{Pe}\Delta\mu^{n+1} = F^{n}$$

$$\mu^{n+1} - \bar{a}\, f^{n+1} + C^2 \Delta f^{n+1} = G^{n}$$

$$F(\mathbf{x},t) = -\mathbf{u}\cdot\nabla f - \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} f - \frac{1}{Pe}\left(\frac{\partial^2\mu}{\partial n^2} + \kappa\frac{\partial\mu}{\partial n}\right),$$

$$G(\mathbf{x},t) = g'(f) - \bar{a} f + C^2\left(\frac{\partial^2 f}{\partial n^2} + \kappa\frac{\partial f}{\partial n}\right),$$

Advantage: stable with large time step $\Delta t = O(\Delta x)$ (could do 2nd order)

Numerical method continued: local level set technique, and enforcing area and surface mass conservation

Local level set method:

computation for the level set function and surfactant concentration are only performed in small tubes around the interface.

Enforcing area conservation: a slightly modified velocity is used for the advection of the level sets to assure total mass flux across the interface is 0

$$\mathbf{u}_h = \tilde{\mathbf{u}}_h + \alpha \mathbf{n},$$

$$\int_\Sigma \mathbf{u}_h \cdot \mathbf{n}\, ds = 0 \quad \Rightarrow \quad \alpha = -\frac{\int_\Sigma \tilde{\mathbf{u}}_h \cdot \mathbf{n}\, \mathrm{d}s}{\int_\Sigma \mathrm{d}s} = -\frac{\int \tilde{\mathbf{u}}_h \cdot \mathbf{n}\delta_\Sigma(\phi)\, \mathrm{d}\mathbf{x}}{\int \delta_\Sigma(\phi)\, \mathrm{d}\mathbf{x}}.$$

Enforcing surfactant conservation: to compensate small numerical diffusion

$$f_h = \beta \tilde{f}_h$$

$$\int_\Sigma f_h(s,t)\, ds = \int_\Sigma f_h(s,0)\, ds \quad \Rightarrow \quad \beta = \frac{\int_{\Sigma_0} f_0\, \mathrm{d}\Sigma_0}{\int_\Sigma \tilde{f}_h\, \mathrm{d}\Sigma} = \frac{\int_\Omega f_0 \delta_{\Sigma_0}\, \mathrm{d}\mathbf{x}}{\int_\Omega \tilde{f}_h \delta_\Sigma\, \mathrm{d}\mathbf{x}}$$

Numerical results

Applied shear flow: $\mathbf{u}_\infty = y\mathbf{e}_y$

Double well potential: $g(f) = f^2(1-f)^2/4$

f=0, f=1 preferred phases.

Surface tension: $\tau(f) = 1 - xf$

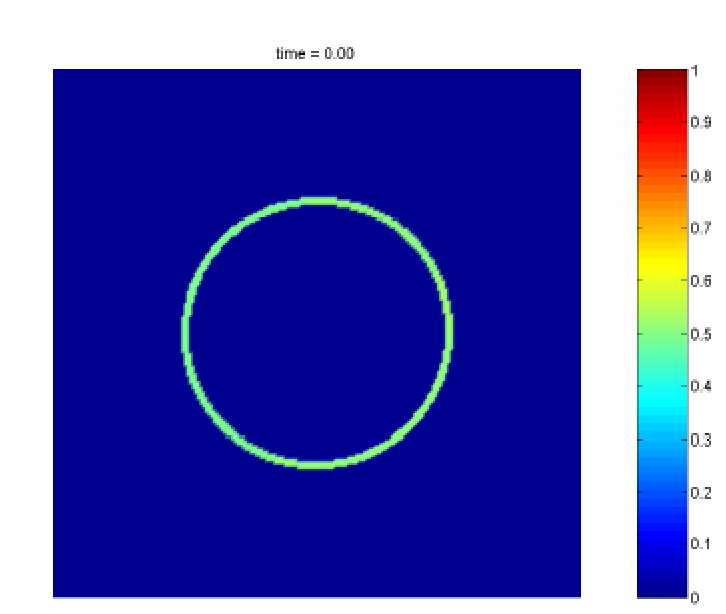
x measures reduction in surface tension of f=1 phase

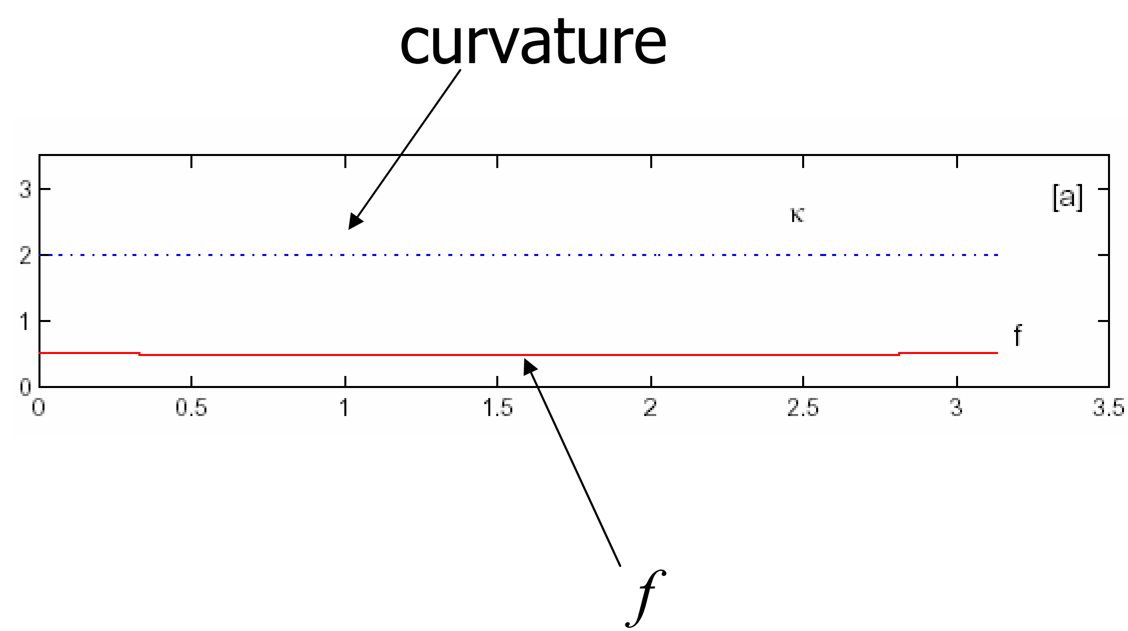
Corresponding surface energy: $\gamma(f) = 1 + xf\log f$

in chem. pot., we take: $\log f \approx 1/2\log(f^2 + C^2)$

Initial condition

$$f(x,y,0) = 0.5 + 0.01(\sin x \cos y + \sin(4x)\cos(3y))$$



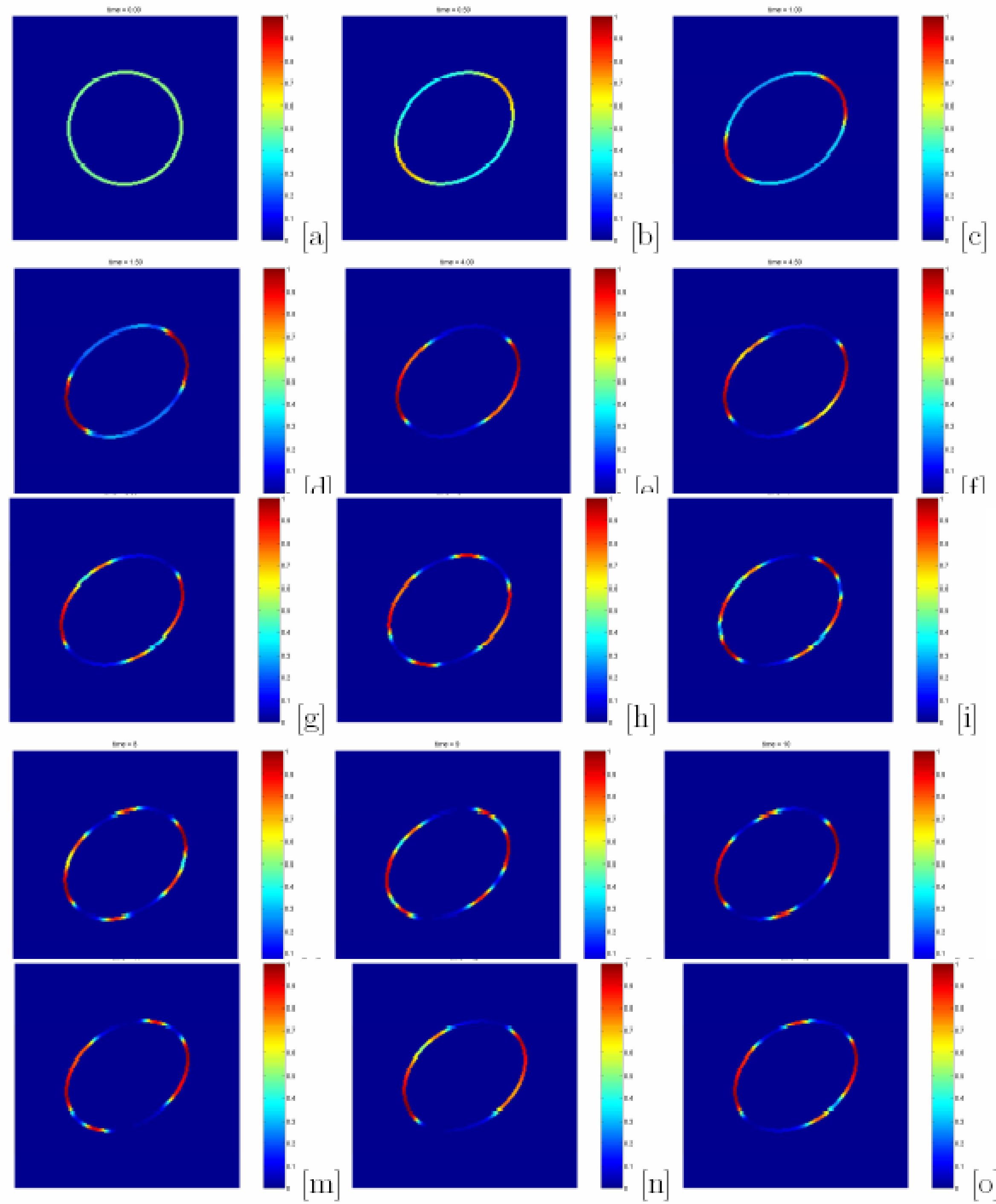


- f perturbed about the spinodal point

Nearly matched surface tension x =0.1

Ca=0.2, Pe=10.
C=0.02, M=1.0

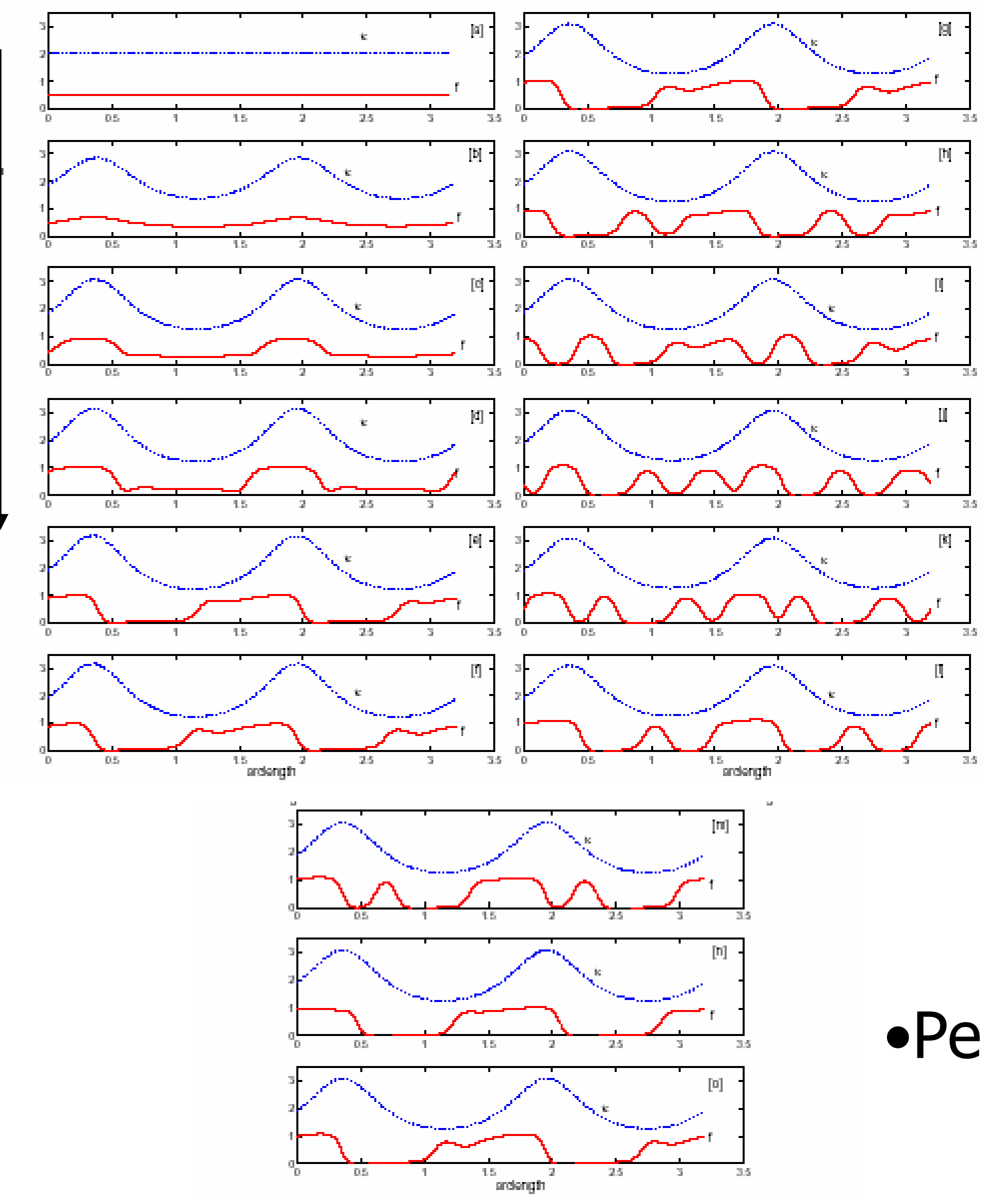
t=0 [a], 0.5 [b], 1.0 [c],
1.5 [d], 4.0 [e], 4.5 [f],
5.0 [g], 6.0 [h], 7.0 [i],
8.0 [j], 9.0 [k], 10.0 [l],
11.0 [m], 12.0 [n], 13.0 [o].



- Surface phase initially decomposes at tips
- Then is swept around drop by the flow

Surface phase distribution

time



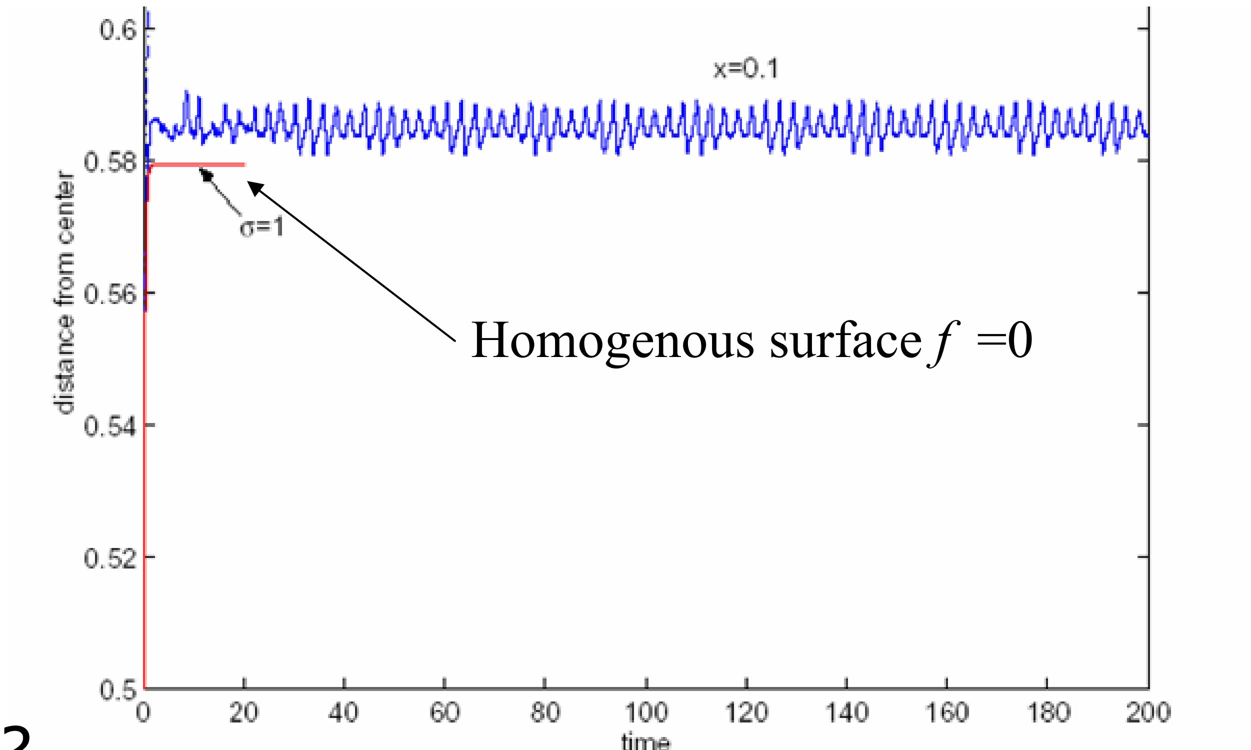
Blue: k

Red: f

•Periodic behavior?

Interface deformation

Maximum distance from center



time

- Periodic?
- Intriguing behavior

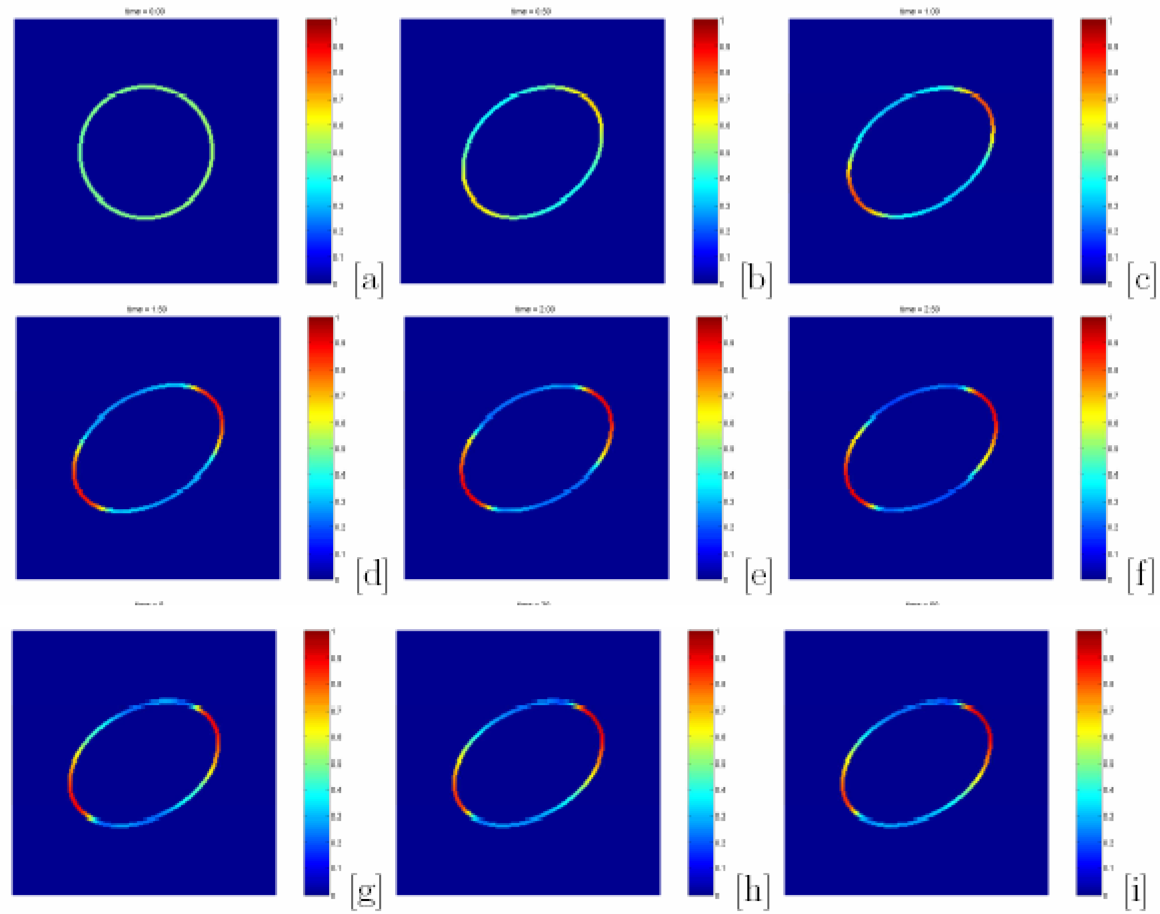
Significantly different surface tensions $x = 0.5$

Ca=0.2, Pe=10.
C=0.02, M=1.0

t=0 [a], 0.5 [b], 1.0 [c],

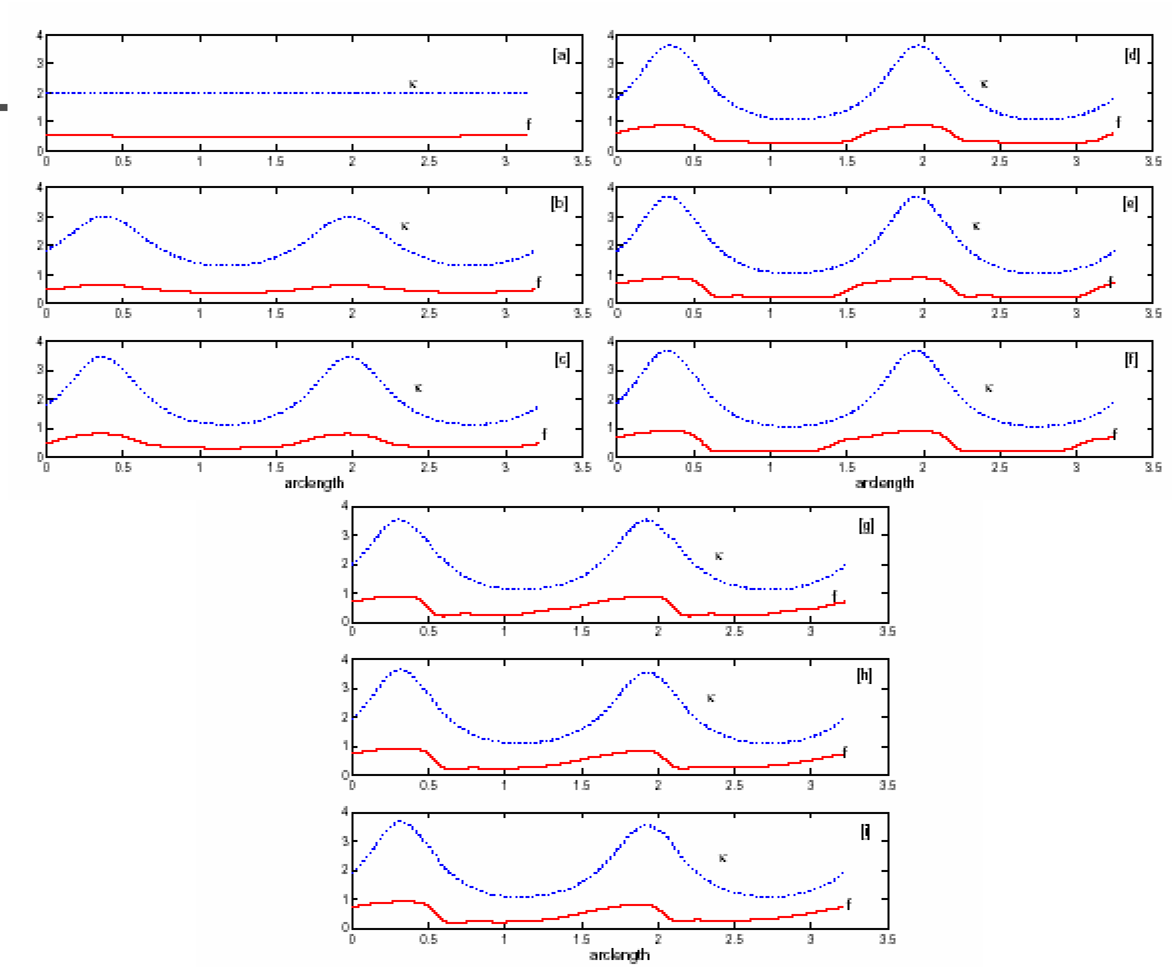
1.5 [d], 2.0 [e], 2.5 [f],

5.0 [g], 30.0 [h], 50.0 [i].



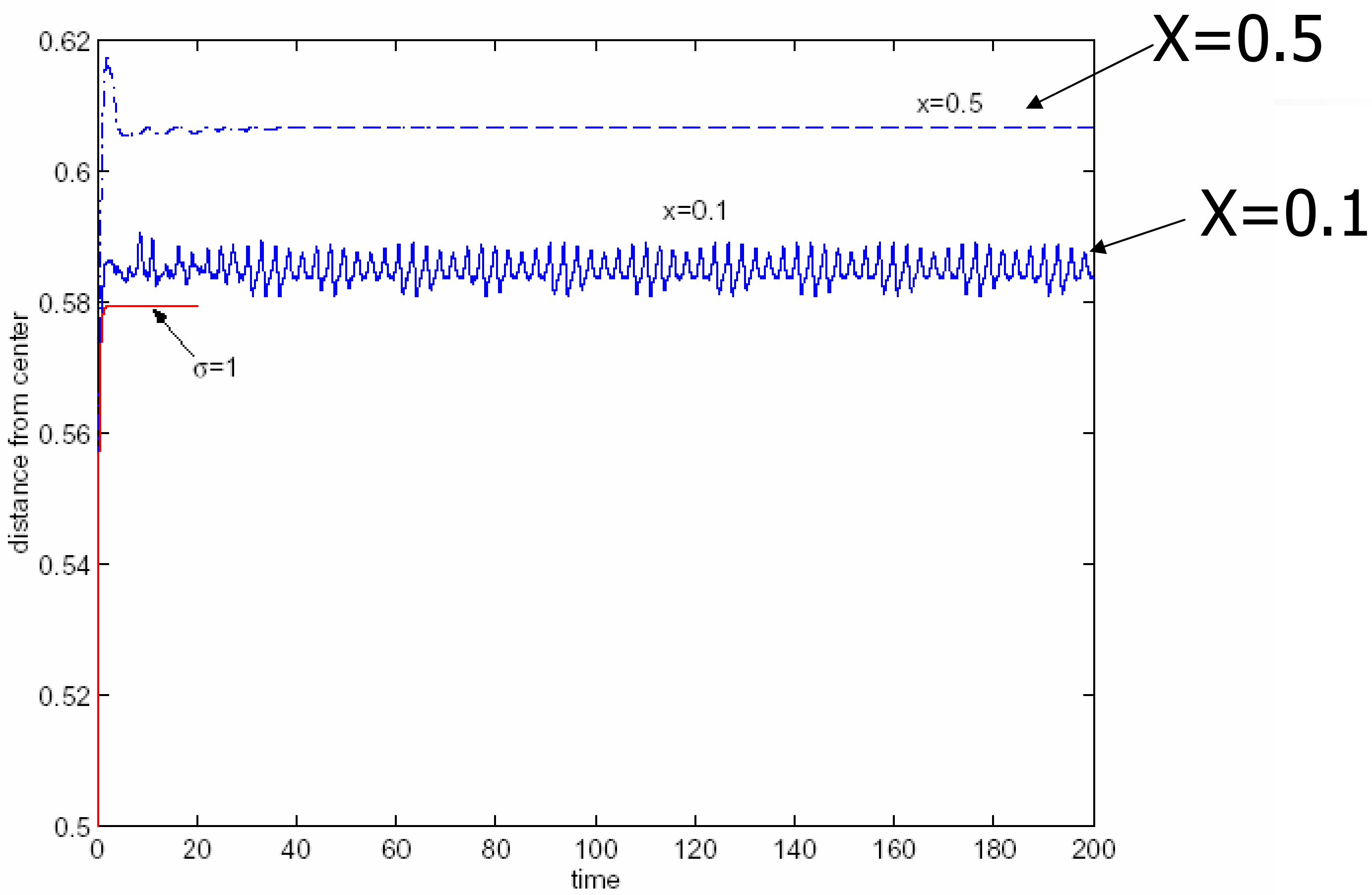
- Phase decomposition occurs at tips
- Steady-state distribution is achieved (energetically favorable to have low surface tension phase at tip)

Surface phase distribution



•Steady-state

Drop deformation



- Deformation larger than for x=0.1 due to smaller surface tension of tip phase (f=1)
- Drop is steady when x=0.5

Conclusions

- Developed general formulation for multicomponent membranes in a fluid flow
- Solved equations in special case of inhomogenous surface Energy and phase-decomposition
- Investigated role of surface tension of phases. Nontrivial behavior. Evidence of periodic solutions when surface tension is nearly matched

Next, incorporate:

- Bending forces
- Inextensibility constraints
- Theory…