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Ray theory for a locally layered random medium

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Abstract. We consider acoustic pulse propagation in inhomogeneous media over relatively long propagation distances. Our main objective is to characterize the spreading of the travelling pulse due to microscale variations in the medium parameters. The pulse is generated by a point source and the medium is modelled by a smooth three-dimensional background that is modulated by stratified random fluctuations. We refer to such media as *locally layered*.

We show that, when the pulse is observed relative to its random arrival time, it *stabilizes* to a shape determined by the slowly varying background convolved with a Gaussian. The width of the Gaussian and the random travel time are determined by the medium parameters along the ray connecting the source and the point of observation. The ray is determined by high-frequency asymptotics (geometrical optics). If we observe the pulse in a deterministic frame moving with the *effective slowness*, it *does not* stabilize and its mean is broader because of the random component of the travel time. The analysis of this phenomenon involves the asymptotic solution of partial differential equations with randomly varying coefficients and is based on a new representation of the field in terms of generalized plane waves that travel in opposite directions relative to the layering.

1. Introduction

When an acoustic pulse propagates through an inhomogeneous medium its shape and travel time are modified by fine-scale heterogeneities. We will analyse in detail this phenomenon and the way in which the modifications depend on the inhomogeneities, which we model as random. Our analysis is partly based on the framework set forth in [1, 25].

The modification of the acoustic pulse is well known in the one-dimensional purely layered case. We consider a generalization to more realistic three-dimensional wave propagation problems, to media that are *locally layered*. Such media have general, three-dimensional, smooth, background variations with a randomly layered microstructure, which need not be plane on the macroscale. The model is motivated by wave propagation in sedimentary rock. Here the sedimentary cycles produce structures that on a microscale might resemble a tilted stack of layers. On top of this local variation there are coarse-scale features that come from macroscopic geological events. As discussed in [21], the motivation for using random media and stochastic equations is the belief that their solution represents physical phenomena which could not be investigated satisfactorily in any other way. In our case a description capturing details of the scattering of the wave by all the heterogeneities of the Earth's crust would be

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prohibitively complex. Moreover, the detailed structure of the Earth's crust is not known. Thus, we replace the actual fine-scale variations by random variations whose statistics reflect those of the actual medium. As a result the propagating pulse becomes a random process. However, up to a random travel time correction, the accumulated effect of the fine-scale layering can be described in a relatively simple, deterministic way. That is, the pulse shape is modified by a convolution in time with a kernel whose parameters depend on a path integral of the medium statistics. The integration path is the geometrical optics characteristic ray path from the surface to the observation point.

The first researchers to describe the modification of the pulse shape where O'Doherty and Anstey in [32]. They did so for a purely layered medium. In section 2 we review some pertinent literature concerning the one-dimensional purely layered case. Then, in section 3 we present our formulation of the more general locally layered problem, the governing equations and the scaling assumptions. The solution to the locally layered problem can be seen as a combination of the high-frequency ray approximation with pulse stabilization in the one-dimensional case. We review the high-frequency approximation in section 4 and discuss the one-dimensional pulse stabilization theory in section 5. The main result is presented in section 6 and it characterizes the modification of the pulse shape and arrival time that is associated with the locally layered medium when the impinging pulse is generated by a point source. In section 7 we derive the approximation, first in the purely layered case by combining the method of stationary phase with an invariant embedding formulation of stochastic boundary value problems. Then we introduce a modification of the high-frequency formulation which enables us to generalize the analysis of the layered case to the locally layered case. Finally, in section 8 we present some concluding remarks.

2. The O'Doherty–Anstey approximation

In order to illustrate our main result concerning pulse propagation in media that are not restricted to layered ones we first consider the simple one-dimensional case with a sound pulse impinging upon a heterogeneous half-space $z > 0$. The half-space $z < 0$ is homogeneous and the pulse impinging on the surface is

$$f = f_0(t/\varepsilon^2).$$

The dimensionless parameter ϵ is small and is introduced so as to distinguish phenomena occurring on different scales. The pressure variations solve for $z > 0$

$$p_{zz} - \gamma^2(z)\, p_{tt} = 0$$

with the slowness $\gamma(z)$, the reciprocal of the local speed of sound, being modelled by

$$\gamma^2(z) = \begin{cases} \gamma_0^2(1 + \varepsilon\, \nu(z/\varepsilon^2)) & \text{for} \quad z \geqslant 0 \\ \gamma_0^2 & \text{for} \quad z < 0. \end{cases} \tag{2.1}$$

The fluctuation ν is a statistically stationary process with rapidly decaying correlations and represents the fine-scale layering. Above we made some important scaling assumptions. First, that the support of the impinging pulse is on the same scale as the fine-scale medium heterogeneity, the scale ε^2. Second, the magnitude of the fluctuations is small, $\mathcal{O}(\varepsilon)$. For short propagation distances the pulse travels essentially undistorted with a speed $1/\gamma_0$. This is described by the effective medium approximation. However, for relatively long propagation distances, an $\mathcal{O}(1)$ distance in the above scaling, the effect of the fine-scale layering becomes appreciable. The scattering associated with the layering gradually delays the pulse and changes its shape, the net effect can be described by

$$\begin{aligned} p(z, \tau_1(z) + \varepsilon\chi_\varepsilon(z) + \varepsilon^2 s) &\sim [f_0(\cdot) \star \mathcal{H}(z, \cdot)](s) \\ &= \int f_0(s - \acute{s})\, \mathcal{H}(z, \acute{s})\, \mathrm{d}\acute{s} \\ &= (1/2\pi) \int \hat{f}_0(\omega)\, \mathrm{e}^{-z\omega^2\gamma_0^2 \bar{l}(\omega)/4}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} \tau_1(z) &= \gamma_0 z \\ \mathcal{H}(z, s) &= (1/2\pi) \int \mathrm{e}^{-z\omega^2\gamma_0^2 \bar{l}(\omega)/4}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \\ \bar{l}(\omega) &= \int_0^\infty C(s)\, \mathrm{e}^{\mathrm{i}\omega 2\gamma_0 s}\, \mathrm{d}s \\ C(s) &= E[\nu(0)\, \nu(s)] \\ \chi_\varepsilon(z) &= \gamma_0 \int_0^z \tfrac{1}{2}\nu(s/\varepsilon^2)\, \mathrm{d}s. \end{aligned}$$

Here τ_1 is the travel time when $\nu \equiv 0$ and $\varepsilon\chi_\varepsilon(z)$ is an $\mathcal{O}(\varepsilon^2)$ random correction. We observe the transmitted pulse in a 'window' of width $\mathcal{O}(\varepsilon^2)$ centred at the corrected travel time. Thus, the time variable s is shifted by $\tau_1 + \varepsilon\chi_\varepsilon(z)$ and scaled by ε^2. The function $\mathcal{H}$ determines the transformation of the pulse shape and is defined in terms of $\bar{l}$ which is the Fourier transform of C, the autocovariance of the random fluctuations ν.

It follows from (2.2) that the pure propagation picture has been modified in two important ways.

First, the travel time to depth z is random and obtained by adding a zero-mean random correction $\varepsilon\chi_\varepsilon$ to $\tau_1(z)$, the travel time associated with the effective medium.

Second, when we observe the pulse relative to a *random arrival time* we see a *deterministic* pulse shape, the original pulse convolved with the deterministic function $\mathcal{H}$. This is what we call *pulse stabilization*. The convolution of the pulse with $\mathcal{H}$ reflects its spreading which is caused by the fine-scale random scattering which in a sense mixes the signal components and causes it to diffuse about its centre.

Note that in the low-frequency limit, when the incident pulse is relatively broad and $\hat{f}_0(\omega)$ is narrowly supported at the origin, then

$$p(z, \tau_1(z) + \varepsilon\chi_\varepsilon(z) + \varepsilon^2 s) \approx [f_0(\cdot) \star \mathcal{N}(z, \cdot)](s) \tag{2.3}$$

$$= (1/2\pi) \int \hat{f}_0(\omega)\, \mathrm{e}^{-z\omega^2\gamma_0^2 l/4}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{2.4}$$

where

$$l = \bar{l}(0) = \int_0^\infty C(s)\, \mathrm{d}s$$

and $\mathcal{N}$ is a centred Gaussian pulse with variance $z\gamma_0^2 l/2$. In this case the pulse is broad relative to the medium fluctuations and does not 'feel' their detailed structure, only the intensity as measured by the correlation length l. If the variance of the process ν is small or its spatial correlation relatively small, the 'effective diffusion' determined by the correlation length l is small, producing less spreading of the pulse.

The study of the effect of fine-scale layering on a propagating pulse was initiated in [32] by O'Doherty and Anstey. On physical grounds they proposed a formula which embodies somewhat implicitly the two effects mentioned above, we will refer to this approximation and its generalizations as the O'Doherty–Anstey pulse-shaping approximation. O'Doherty and Anstey based their derivations on a discrete equal travel time representation of the medium. The first researchers to give a mathematical account for the phenomenon in the continuous case were Banik *et al* in [3, 4]. They obtained the O'Doherty–Anstey approximation using a mean-field approach, and applied it to investigate the pulse shaping associated with specific stochastic models for the heterogeneity. Resnick *et al* [34] present an interesting alternative derivation of the formula, and were the first to approach the problem from an invariant embedding point of view. However, the first rigorous account for the stabilization phenomenon was given by Burridge *et al* in [11]. Here they derive the version of the formula which applies to an equal travel time discretized medium using an averaging technique. Based on this result Burridge *et al* performed a careful numerical investigation in [7, 8], which showed that the formula generalizes to elastic wave propagation with obliquely travelling plane waves. Moreover, with this as their starting point they also generalize the formula to pulses generated by a point source over a layered medium in [9] by decomposing the source in terms of plane waves and using a stationary phase argument. Berlyand and Burridge [5] derive a correction estimate for the O'Doherty–Anstey approximation based on an equal travel time or Goupillaud representation of the medium. Asch *et al* [1] presents the first rigorous derivation of the formula in a continuous framework using invariant embedding and by applying a limit theorem for stochastic ordinary differential equations. This analysis was generalized to reflections, rather than only the directly transmitted pulse, in [33] by Papanicolaou and Lewicki. Finally, in [27] Lewicki generalizes the O'Doherty–Anstey formula to certain rather general hyperbolic systems using an averaging approach.

In these last three reports the fluctuations in the medium were assumed to be differentiable. Furthermore, in all of the above the fluctuations were assumed to be weak as in (2.1). The fact that the coupling to the fluctuations is weak is what allows one to probe the medium with a pulse on the same scale as the fluctuations, the scale ε^{-2}, and still observe stabilization. Recently, a different type of medium model has been considered. Here, the fluctuations are strong $\mathcal{O}(1)$ and not necessarily continuous with the slowness being modelled by

$$\gamma^2(z) = \gamma_0^2(1 + \nu(z/\varepsilon^2)).$$

In this scaling the impinging pulse is $f_0(t/\varepsilon)$ and thus is defined on the time scale t/ε rather than t/ε^2, otherwise the pulse would interact strongly with every feature of the random medium and a characterization of the transmitted pulse in general terms would not be possible. What enables one to push through the argument showing stabilization is the rapid variations in the fluctuations. For this new scaling of the source there would be no pulse shaping associated with weak fluctuations. For strong fluctuations the description (2.3) essentially prevails, now

$$p(z, \tau_1(z) + \chi_\varepsilon + \varepsilon s) \sim [f_0(\cdot) \star \mathcal{N}(z, \cdot)](s) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{2.5}$$

with $\mathcal{N}$ defined as in (2.3), it is a centred Gaussian pulse. As above we observe the transmitted pulse in a randomly corrected time frame and on the scale of the incident pulse width, in this case the $\mathcal{O}(\varepsilon)$ scale. The variance V of $\mathcal{N}$ and the random travel time correction χ_ε to the effective medium travel time τ_1 are given by

$$\begin{aligned} V(z) &= \int_0^z \tfrac{1}{2}\gamma_0^2 l \, \mathrm{d}s = z\gamma_0^2 l/2 \\ \chi_\varepsilon(z) &= \gamma_0 \int_0^z \tfrac{1}{2}\nu(s/\varepsilon^2)\, \mathrm{d}s \\ \tau_1(z) &= \int_0^z \gamma_0 \, \mathrm{d}s = \gamma_0 z. \end{aligned} \tag{2.6}$$

This model was first analysed by Burridge *et al* in [10] and Clouet and Fouque in [14] using invariant embedding approaches. In [28] Lewicki *et al* generalize the results to certain rather general hyperbolic systems. In [13] Chillan and Fouque extend the theory to the case of a point source over a strongly heterogeneous layered half-space. It is important to note that the above description of the transmitted pulse is in a *random* time frame. The 'coherent' field, or the mean pulse is in a *deterministic* time frame and will be broader since it is obtained by 'averaging' pulses that are dispersed by the random travel time correction χ_ε. As described in detail in section 5.3, the broadening of the pulse in a deterministic frame is approximately *double* that of the broadening in the random frame.

O'Doherty and Anstey's original paper was motivated by the need to characterize the effect of fine-scale heterogeneity in seismic wave propagation. Realizing the importance of such a description a string of studies followed, aimed at a more analytical derivation of the formula and at extending it to more general medium models. However, these deal with purely *layered* media. To be able to describe wave propagation in an actual application it is necessary to understand the significance of lateral variation in the parameters. Here we present a theory [36] that generalizes the O'Doherty Anstey approximation to locally layered media. Recently, in [26], the O'Doherty–Anstey approximation and also a theory for the signal fluctuations was derived for a locally layered medium when the lateral variations are small. We show in detail that the results we present here specialize to the O'Doherty–Anstey results presented in [26]. The model that we use in this paper has general three-dimensional smooth, deterministic background variations which are modulated by stratified fine-scale random fluctuations. We base the analysis on a new representation of the wave field in terms of locally up- and down-propagating waves and a new way of describing their interaction.

We find that for locally layered media the main aspects of the layered results prevail. The random travel-time correction and the kernel modifying the pulse shape are as in (2.5). However, now the depth variable z is replaced by the arc-length parameter along the geometrical optics ray path that connects the source to the observation point and is associated with solving the eiconal equation for the deterministic part of the medium. The parameters in (2.5) are, in the general case, defined essentially as in (2.6), only the path of integration becomes the geometrical optics ray. A precise statement of this result is given in section 6.

The spreading of pulses propagating in random media arises in many contexts, in astrophysics, in radar, in underwater sound propagation and elsewhere, and has received a lot of attention [40, 41], and more recently [39]. In these studies the random inhomogeneities are isotropic. They are not layered or nearly layered as in our study which is motivated by geophysical applications. The methods used in the isotropic case are mostly based on the parabolic or paraxial approximation for time harmonic waves, and the associated moment equations, followed by a Fourier synthesis. The connection with geometrical optics can also be made through path integrals [15, 18, 39, 41] and their asymptotic analysis. There is no random centring in time to be made in the isotopic case, so the phenomenon of *pulse stabilization* does not arise in the form that it does for randomly layered media. The possibility of having some kind of pulse stabilization in isotropic random media is not precluded but there is no clear experimental or theoretical evidence for it at this point.

The *mean* field or ensemble average of the propagated pulse is also investigated in the early papers [20, 23], in the case with weak random fluctuations.

Another approximation that is valid for relatively long propagation distances and weak heterogeneities is the so-called Markov approximation. This amounts to letting the inhomogeneities be δ-correlated in the direction of propagation which leads to closed moment equations [16]. This approximation can be combined with the parabolic approximation [2] and give results about the statistical character of the fluctuations in the wave field [19].

Our primary focus in this paper is the pulse stabilization which gives a description of the pulse itself, for a single realization of the random medium.

3. The locally layered model equations

We consider acoustic wave propagation in three-space dimensions. Let $\boldsymbol{u}(\boldsymbol{x}, z, t)$ and $p(\boldsymbol{x}, z, t)$ be the acoustic velocity and pressure satisfying the equation of continuity of momentum and mass

$$\begin{aligned} &\rho \boldsymbol{u}_t + \nabla p = \boldsymbol{F}_\varepsilon(\boldsymbol{x}, z, t) \\ &K_\varepsilon^{-1}(\boldsymbol{x}, z)\, p_t + \nabla \cdot \boldsymbol{u} = 0 \end{aligned} \tag{3.1}$$

where t is time, z is the depth into the medium and $\boldsymbol{x} = (x_1, x_2)$ are the horizontal coordinates. Note that z is defined so as to increase with depth. Furthermore, ρ and K_ε^{-1} are material properties, density and compliance, respectively. Above and in the following boldface indicates a vectorial quantity. The geometry of the problem is shown in figure 1. A point source, modelled by $\boldsymbol{F}_\varepsilon$, is located in the homogeneous half-space $z < 0$ and initiates a pulse impinging on the heterogeneous half-space. Our main objective is to characterize how the heterogeneities transform the pulse as it travels. To leading order this amounts to identifying the convolving function indicated by a question mark in figure 1. Recall that ε is a small dimensionless parameter separating the different scales in the problem. The compliance, K_ε^{-1}, has a two-scale structure. Its mean varies on a macroscopic scale while it is being randomly modulated on a microscopic scale. We find it convenient to order the various length scales relative to the macroscopic scale of the compliance corresponding to the macroscopic propagation distance which is an $\mathcal{O}(1)$ quantity.

We will consider two qualitatively different models for the character of the material properties, corresponding to two different choices in the definition of K_ε^{-1} in terms of ε.

First, we consider what we refer to as a *locally layered strongly heterogeneous* random medium. In this case the material properties, density and compliance are modelled by

$$\begin{aligned} &\rho(\boldsymbol{x}, z) \equiv \rho_0 \\ &K_\varepsilon^{-1}(\boldsymbol{x}, z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(\boldsymbol{x}, z)(1 + \nu(\Phi(\boldsymbol{x}, z)/\varepsilon^2)) & z \in (0, \infty) \end{cases} \end{aligned} \tag{3.2}$$

where the mean K_1^{-1} is a *smooth* and positive function. The fluctuation ν modulating the compliance is a zero-mean, stationary stochastic process bounded from below by $(-1 + d)$, with d a positive constant. It is assumed to have a rapidly decaying correlation function. Note that the random modulation includes the smooth function Φ. This model can be transformed, by a change of variables as in appendix C of [38], to the more special one in which the modulation term is a function of depth only. Hence, in the following we assume $\nu = \nu(z/\varepsilon^2)$.

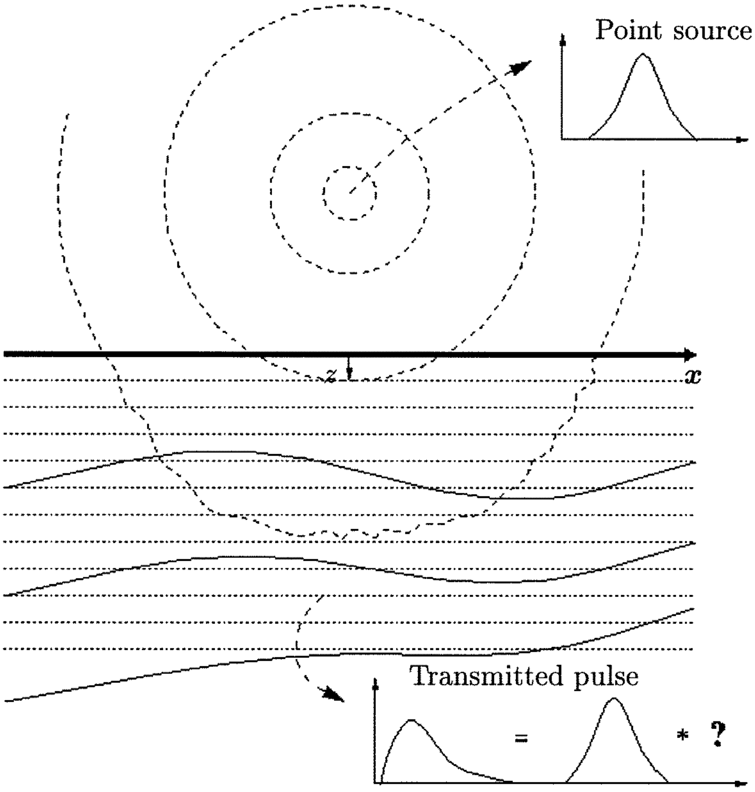

Figure 1. The physical problem that we consider. A point source is located in a homogeneous half-space and generates a spherical wave that is impinging on the heterogeneous locally layered half-space. The locally layered medium is comprised of a smooth three-dimensional background randomly modulated by stratified microscale variations. We are interested in the shape of the transmitted pulse after it has propagated through the random medium.

The fluctuation embodies the random character of the medium. The forcing is due to the point source

$$\boldsymbol{F}_\varepsilon(\boldsymbol{x}, z, t) = \varepsilon f(t/\varepsilon)\,\delta(\boldsymbol{x})\,\delta(z - z_s)\,\boldsymbol{e} \tag{3.3}$$

where f is the pulse shape, $\boldsymbol{e}$ is the source directivity vector and the source location is $(\mathbf{0}, z_s)$, with $z_s < 0$. In order to simplify the formulae we will assume a vertical source $\boldsymbol{e} = (\mathbf{0}, 1)'$ and the matched medium case, that is $K_1^{-1}(\boldsymbol{x}, 0) \equiv K_0^{-1}$. The ε scaling of the source magnitude has been introduced only to make the transmitted pulse an $\mathcal{O}(1)$ quantity.

What sets the above model apart from previously considered models is that the mean compliance K_1^{-1} is a function of all space coordinates. Furthermore, the fluctuation process ν is a function of the level surfaces of Φ rather than depth only. On the finest scale of the model, the scale of the fluctuations, the medium variations are essentially one dimensional. We therefore refer to the model as locally layered. The rationale for denoting the model as strongly heterogeneous is that the amplitude of the random modulation is $\mathcal{O}(1)$, and not small. Note also that the source is defined on a scale intermediate between that of the fluctuations and that of the background medium. In this scaling the effect of the macroscopic features of the medium on the propagating wave can be analysed by a high-frequency approximation, while the propagation relative to the microstructure can be understood in terms of averaging of stochastic equations.

Second, we consider a scaling in which the density and compliance are modelled by

$$\rho(\boldsymbol{x}, z) \equiv \rho_0$$
$$K_\varepsilon^{-1}(\boldsymbol{x}, z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(\boldsymbol{x}, z)(1 + \varepsilon\nu(\Phi(\boldsymbol{x}, z)/\varepsilon^2)) & z \in (0, \infty) \end{cases} \tag{3.4}$$

and refer to this as a *locally layered weakly heterogeneous* random medium. In this case the source is taken to be

$$\boldsymbol{F}_\varepsilon(\boldsymbol{x}, z, t) = \varepsilon^2 f(t/\varepsilon^2)\,\delta(\boldsymbol{x})\,\delta(z - z_s)\,\boldsymbol{e}. \tag{3.5}$$

This model differs from (3.2) and (3.3) only in that the amplitude of the random fluctuations is $\mathcal{O}(\varepsilon)$, and that the support of the source is on the same scale as the fluctuations, the scale ε^2. This is the appropriate scaling in the weakly heterogeneous case when the coupling between the propagating pulse and the random process ν is weak.

In both of the above cases we assume that the medium is initially at rest

$$\left.\begin{aligned} f(t) &= 0 \\ p(\boldsymbol{x}, z, t) &= 0 \\ \boldsymbol{u}(\boldsymbol{x}, z, t) &= \boldsymbol{0}. \end{aligned}\right\} \quad \text{for} \quad t \in (-\infty, 0] \tag{3.6}$$

When the mean compliance K_1^{-1} and the fluctuations ν are functions of depth, z, only, then the random medium is purely layered and this situation has been studied extensively (see, for example, [1, 14, 33]).

The pulse-shaping results can also be generalized to the case with random variations in the density ρ. Here, for ease of presentation, we will deal exclusively with the models defined by (3.1)–(3.3) and (3.4), (3.5) and the layered versions thereof. We assume $\Phi(\boldsymbol{x}, z) = z$ and the case with general Φ is discussed in appendix C of [36, 38].

The strongly heterogeneous model (3.2) is relevant for instance in the context of reflection seismology, in a region with strong variations in the earth parameters. Then the incident pulse is typically about 50 m wide, which is large relative to the *strong* (in amplitude) fine-scale medium fluctuations which are on the scale of metres, but small relative to the distance travelled by the pulse, which is on the scale of kilometres.

The weakly heterogeneous model (3.4) is relevant when the variations in the earth parameters are small [37]. It is also relevant in surface gravity waves in the long-wavelength regime [30], or in the context of sound pulses propagating in a fluctuating ocean. In the latter case the fine-scale variations occur typically on the scale of kilometres, the scale of internal waves, and are *weak*. The propagation length is typically of the order of 100 km.

4. High-frequency asymptotics

When there are no fluctuations, $\nu \equiv 0$, we can analyse (3.1) in the high-frequency or geometrical optics approximation [22]. In this section we review the geometrical optics approximation for the deterministic case since this motivates our approach in later sections. The approximation for the transmitted pulse in the random case is a modification of the deterministic one. We carry out the calculations for the strongly heterogeneous model. The result in the weakly heterogeneous case is completely analogous. The deterministic case corresponds to using effective medium parameters, appropriate for short propagation distances, on the order of several pulse lengths [29]

We consider first the simple one-dimensional version of the problem.

4.1. Approximation for a one-dimensional medium

In the one-dimensional case a normally incident plane wave is impinging upon a purely layered medium. The governing equations with appropriate scaling in the deterministic case are

$$\begin{aligned} \rho u_t + p_z &= f(t/\varepsilon)\,\delta(z - z_s) \\ K_1^{-1}(z)\, p_t + u_z &= 0. \end{aligned} \tag{4.1}$$

Define the scaled Fourier transform by

$$\hat{p}(z, \omega) = \int p(z, s)\, \mathrm{e}^{\mathrm{i}\omega s/\varepsilon}\, \mathrm{d}s.$$

From (4.1) it follows that for $z > 0$

$$\hat{p}_{zz} + (\omega/\varepsilon)^2\, \gamma_1^2\, \hat{p} = 0. \tag{4.2}$$

The slowness γ_1 is the effective medium slowness [29] in the locally layered medium

$$\gamma_1(z) = \sqrt{E[K_1^{-1}(z)(1 + \nu(z/\varepsilon^2))]\,\rho_0} = \sqrt{K_1^{-1}(z)\,\rho_0}. \tag{4.3}$$

The slowness is a constant, γ_0, for $z < 0$. We assume a matched boundary condition, that is $\gamma_1(0) = \gamma_0$. The leading-order high-frequency approximation now amounts to an approximation for $\hat{p}$ of the form

$$\hat{p}(z, \omega) \sim A(z, \omega)\, \mathrm{e}^{\mathrm{i}\omega\tau_1(z)/\varepsilon} \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{4.4}$$

and then requiring (4.2) to be satisfied to first order. We find in the usual way that

$$\begin{aligned} \tau_1(z) &= \int_{z_s}^{z} \gamma_1(s)\, \mathrm{d}s \\ A(z, \omega) &= A(0, \omega)\sqrt{\gamma_0/\gamma_1(z)} \end{aligned} \tag{4.5}$$

where the phase $\tau_1(z)$ is the effective medium travel time to depth z. Note that the pulse impinging on the heterogeneous half-space, $z > 0$, does not depend on the value of γ_1 in this half-space. We can therefore find $A(0, \omega)$ by comparison with the purely homogeneous case when $\gamma(z) \equiv \gamma_0$. Upon back-transforming (4.4) in time and substituting the value for $A(0, \omega)$, we obtain

$$p(z, \tau_1(z) + \varepsilon s) \sim 2^{-1}\sqrt{\gamma_0/\gamma_1(z)}\, f(s) \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{4.6}$$

Thus, the travel time τ_1 provides a centring and the amplitude term scales the pulse in terms of variations in the slowness. For $z > 0$ the high-frequency approximation expresses the propagating wave in terms of a down-travelling wave mode only. In the simplest case discussed above, the approximation is just a translation of the source pulse scaled by a geometric factor. We show next that for wave propagation in three dimensions the geometric factor also reflects dispersion and confluence of the characteristic rays associated with the wave, moreover, the phase then represents the travel time along these rays.

4.2. Propagation from a point source

Consider the deterministic version of the strongly heterogeneous model defined in (3.1)–(3.3) and (3.6). In view of the form of the incident pulse, we define the scaled Fourier transform as

$$\hat{p}(\boldsymbol{x}, z, \omega) = \int p(\boldsymbol{x}, z, s)\, \mathrm{e}^{\mathrm{i}\omega s/\varepsilon}\, \mathrm{d}s.$$

By elimination of $\boldsymbol{u}$ in (3.1) we find that the time-transformed pressure solves, in the effective medium/deterministic case,

$$\Delta \hat{p} + (\omega/\varepsilon)^2 \gamma_1^2\, \hat{p} = \varepsilon^2 \hat{f}(\omega)\, \delta(\boldsymbol{x})\, \delta'(z - z_s) \tag{4.7}$$

where the effective medium slowness is

$$\gamma_1(\boldsymbol{x}, z) = \sqrt{E[K_\varepsilon^{-1}(\boldsymbol{x}, z)]\, \rho_0} = \sqrt{K_1^{-1}(\boldsymbol{x}, z)\, \rho_0} \tag{4.8}$$

for $z > 0$. The upper half-space is homogeneous with slowness γ_0. We assume again a matched boundary condition, that is $\gamma(\boldsymbol{x}, 0) = \gamma_0$. The high-frequency approximation for the point source problem associated with the reduced wave equation

$$\mathcal{L}u \equiv \Delta u + (\omega/\varepsilon)^2\, \gamma_1^2 u = -\delta(\boldsymbol{x})\, \delta(z - z_s) \tag{4.9}$$

has the form

$$u \sim A\, \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon}. \tag{4.10}$$

The phase τ_1 is the travel time to a given point in the medium. The amplitude A describes how the pulse spreads as it travels. Substituting (4.10) in (4.9) we find that away from the source the phase, τ_1, solves the eiconal equation

$$(\nabla\tau_1)^2 = \begin{cases} \gamma_0^2 & \text{for} \quad z < 0 \\ \gamma_1^2(\boldsymbol{x}, z) & \text{for} \quad z \geqslant 0 \end{cases} \tag{4.11}$$

and A satisfies

$$2\nabla\tau_1 \cdot \nabla A + \Delta\tau_1\, A - \mathrm{i}\varepsilon/\omega \Delta A = 0. \tag{4.12}$$

The leading-order approximation for A is obtained by requiring it to solve the first-order transport equation, that is

$$2\nabla\tau_1 \cdot \nabla A_0 + \Delta\tau_1 A_0 = 0. \tag{4.13}$$

In order to obtain correct initial conditions we consider (4.9) in a neighbourhood of the source and match the approximation with the free-space Green's function of the homogeneous case, with the homogeneous parameters equal to those at the source point. Thus, we choose $\tau_1 = 0$ at the source and increasing isotropically away from it. The eiconal equation can be solved by the method of characteristics and the transport equation as an ordinary differential equation integrated along the characteristics. An example showing a possible configuration of characteristic rays orthogonal to the phase fronts $\tau_1 =$ constant is shown in figure 2. Making use of the source point condition to find the initial value for the amplitude at the source point we arrive at the following approximation for u:

$$u \sim \frac{\sqrt{(\mathrm{d}\Omega/\mathrm{d}a)(\gamma_0/\gamma_1)}}{4\pi}\, \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} \qquad \text{as} \quad \varepsilon \downarrow 0$$

as in (8.10) of [22]. Here γ_0 is the slowness at the source point $(\mathbf{0}, z_s)$ and $\mathrm{d}\Omega$ is an element of solid angle of the initial directions of rays about the ray path passing through $(\boldsymbol{x}, z)$ and $\mathrm{d}a$ is the associated element of area on the phase front. The ray passing through $(\boldsymbol{x}, z)$ is denoted Γ and shown by a full curve in figure 2.

Since

$$\mathcal{L}[-\varepsilon^2 \hat{f}\, \partial_z u] = \varepsilon^2 \hat{f}(\omega)\, \delta(\boldsymbol{x})\, \delta'(z - z_s)$$

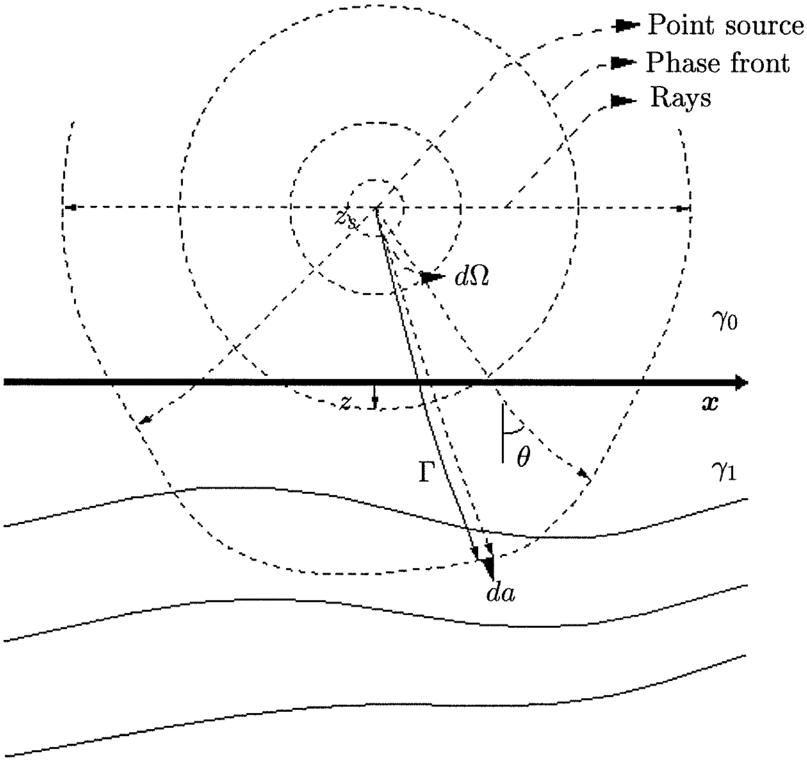

Figure 2. Rays associated with propagation from a point source. In the upper half-space the slowness γ_0 is constant, then the rays are straight lines and the phase fronts that are orthogonal to these are spherical. In the lower half-space the medium is not homogeneous leading to a curved ray geometry and more general phase fronts.

the leading-order asymptotic approximation for p is

$$p(\boldsymbol{x}, z, \tau_1 + \varepsilon s) \sim \frac{\tau_{1,z}\sqrt{(\mathrm{d}\Omega/\mathrm{d}a)(\gamma_0/\gamma_1)}}{4\pi} f'(s) \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{4.14}$$

In the special case that the medium is homogeneous, with a constant slowness γ_0, the above approximation becomes

$$p(\boldsymbol{x}, z, \tau_1 + \varepsilon s) \sim \frac{\gamma_0 \cos(\theta)}{4\pi r} f'(s) \qquad \text{as} \quad \varepsilon \downarrow 0$$

with r the distance from the source to the observation point $(\boldsymbol{x}, z)$. In this case the problem can be solved explicitly and the exact expression for the transmitted pressure is

$$G_f = \frac{\gamma_0 \cos(\theta)}{4\pi r} f'(s) + \frac{\varepsilon \cos(\theta)}{4\pi r^2} f(s). \tag{4.15}$$

In the following we need some assumptions about the phase τ_1 associated with the slowness γ_1. Let Γ denote, as above, the characteristic ray segment from the source point to the point of observation. We assume that the path Γ is nowhere horizontal and that τ_1 is a uniquely defined smooth function in a neighbourhood of this path.

For long propagation distances, scattering by the random fluctuations in the medium parameters causes an appreciable statistical coupling between up- and down-travelling modes.

In order to account for this coupling we shall modify the ansatz (4.10) so as to include an up-travelling wave mode as well as a down-travelling wave mode and show how the O'Doherty–Anstey approximation comes about from this coupling. First, we state the results, the O'Doherty–Anstey pulse-shaping approximation in the one-dimensional case and how this generalizes to locally layered media.

5. The layered pulse-shaping approximation

We present the pulse-shaping approximation in the one-dimensional case, that is, for a strictly layered medium with an normally incident plane wave. In this case the approximation is well known [1, 10, 14]. Our main result, the pulse-shaping approximation in a locally layered medium, is given in the next section. We review here the approximation in the one-dimensional case, since the general result can be interpreted as a combination of this and the geometrical optics approximation discussed above. In section 5.2.2 we give a new interpretation of the pulse-shaping approximation for weakly heterogeneous media in terms of the distribution of a random sum. Note that we do not require the weakly heterogeneous medium to be differentiable as in [1].

5.1. Strong medium fluctuations

We consider first the results in the strongly heterogeneous case. A normally incident plane-wave pulse is impinging upon the layered half-space $z > 0$ and the governing equations are

$$\begin{aligned} &\rho u_t + p_z = F_\varepsilon(z,t) \\ &K_\varepsilon^{-1}(z)\, p_t + u_z = 0 \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} &\rho(z) \equiv \rho_0 \\ &K_\varepsilon^{-1}(z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(z)(1+\nu(z/\varepsilon^2)) & z \in (0,\infty) \end{cases} \\ &F_\varepsilon(z,t) = f(t/\varepsilon)\,\delta(z - z_s). \end{aligned}$$

Define the (scaled) Fourier transform of the pressure pulse by

$$\hat{p}(z,\omega) = \int p(z,s)\, e^{i\omega s/\varepsilon}\, ds.$$

We show in section 7.1 that

$$E[\hat{p}(z,\omega)\, e^{-i\omega\chi_\varepsilon/\varepsilon}] \sim 2^{-1}\sqrt{\gamma_0/\gamma_1(z)}\, \hat{f}(\omega) \exp\left[-\omega^2 \int_0^z \gamma_1^2(s)(l/4)\, ds\right] e^{i\omega\tau_1/\varepsilon} \quad \text{as} \quad \varepsilon \downarrow 0 \tag{5.2}$$

with

$$\begin{aligned} &\gamma_1(z) = \sqrt{\rho_0/K_1(z)} \\ &l \equiv \int_0^\infty E[\nu(0)\,\nu(s)]\, ds \\ &\tau_1(z) = \int_{z_s}^z \gamma_1(s)\, ds \\ &\chi_\varepsilon(z) = \int_0^z \tfrac{1}{2}\gamma_1(s)\,\nu(s/\varepsilon^2)\, ds. \end{aligned} \tag{5.3}$$

The slowness γ_1 is the reciprocal of the local speed of sound for the *effective* [29] medium and the correlation length l is a measure of the strength and coherence of the random fluctuations. In view of (4.6) we find that when corrected for a random phase the mean of the time harmonic amplitude is simply the amplitude in the deterministic case multiplied by a Gaussian function. This simple description concerns, however, only the *mean*. The time harmonic amplitude itself exhibits strong random fluctuations. It is interesting, however, that when we transform (5.2) to the time domain we obtain a *pulse* whose random fluctuations have disappeared. If we 'open a time window at the random arrival time' by 'centring' with respect to the random phase and using the time scaling of the source, we find

$$p(z, \tau_1 + \chi_\varepsilon + \varepsilon s) \sim [G_f \star \mathcal{N}](s) = \int G_f(z, s - \acute{s})\,\mathcal{N}(z, \acute{s})\,\mathrm{d}\acute{s}$$
$$= (1/2\pi) \int \hat{G}_f(z, \omega)\, \exp\left[-\omega^2 \int_0^z \gamma_1^2(u)(l/4)\,\mathrm{d}u\right] \mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{5.4}$$

where $G_f(z, (t - \tau_1)/\varepsilon)$ is the exact transmitted pressure pulse in the deterministic case when $\nu \equiv 0$. The high-frequency approximation for G_f is given by (4.6). In the above frame the transmitted pulse appears simply as that of the deterministic medium convolved with a pulse-shaping function $\mathcal{N}$ that solves

$$\mathcal{N}_z = D(z)\,\mathcal{N}_{ss} \tag{5.5}$$

$$\mathcal{N}(0, s) = \delta(s) \tag{5.6}$$

with

$$D(z) = \tfrac{1}{4} l \gamma_1^2(z) \tag{5.7}$$

that is, a diffusion equation with depth playing the role of time. Thus, $\mathcal{N}$ is a Gaussian pulse of squared width $2\int_0^z D(s)\,\mathrm{d}s = \frac{1}{2} l \int_0^z \gamma_1^2(s)\,\mathrm{d}s$. This is the O'Doherty–Anstey pulse-shaping approximation first derived in [10, 14]. The transmitted pulse of the deterministic medium has been modified in two ways. First, the arrival time contains a small random component. Second, when observed in the time frame defined by the random arrival, the pulse shape is to leading order a *deterministic* modification of the one in the effective medium. The modification is through convolution with a Gaussian pulse-shaping function.

5.2. *Weak medium fluctuations*

Next, we consider the one-dimensional weakly heterogeneous case with a normally incident plane-wave pulse impinging upon the layered half-space $z > 0$. The governing equations are still (5.1), but now

$$\begin{aligned} &\rho(z) \equiv \rho_0 \\ &K_\varepsilon^{-1}(z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(z)(1 + \varepsilon\nu(z/\varepsilon^2)) & z \in (0, \infty) \end{cases} \\ &F_\varepsilon(z, t) = f(t/\varepsilon^2)\,\delta(z - z_s). \end{aligned} \tag{5.8}$$

The model differs from the one above in that the medium fluctuations are weak $\mathcal{O}(\varepsilon)$ and that the source pulse is supported on the scale of the fluctuations, the scale ε^2. In this case we find

$$p(z, \tau_1 + \varepsilon\chi_\varepsilon + \varepsilon^2 s) \sim [G_f \star \mathcal{H}](s) = \int G_f(z, s - \acute{s})\, \mathcal{H}(z, \acute{s})\, \mathrm{d}\acute{s}$$

$$= (1/2\pi) \int \hat{G}_f(z, \omega) \exp\left[-\omega^2 \int_0^z \tfrac{1}{4}\gamma_1^2(u)\, \bar{l}(u, \omega)\, \mathrm{d}u\right] \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{5.9}$$

with

$$\mathcal{H}(z, s) = 1/(2\pi) \int \exp\left[-\omega^2 \int_0^z \tfrac{1}{4}\gamma_1^2(u)\, \bar{l}(u, \omega)\, \mathrm{d}u\right] \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega$$

$$\bar{l} = \bar{l}(z, \omega) = \int_0^\infty C(u)\, \mathrm{e}^{\mathrm{i}\omega 2\gamma_1(z) u}\, \mathrm{d}u$$

$$C(u) = E[\nu(0)\, \nu(u)]$$

where $G_f(z, (t - \tau_1)/\varepsilon^2)$ is the exact transmitted pressure in the deterministic case and the other quantities are defined as in the previous section. As above, in the *time frame* defined relative to the random arrival the pressure p is described asymptotically as a *deterministic* pulse that is obtained by convolving the pulse in the effective medium with the pulse-shaping function $\mathcal{H}$. The approximation (5.9) was first derived in [1]. The pulse-shaping function solves the differential equation

$$\begin{aligned} &\partial_z \mathcal{H}(z, s) = (\gamma_1(z)/8)\partial_s^2[h(z, \cdot) \star \mathcal{H}(z, \cdot)](s) \\ &\mathcal{H}(0, s) = \delta(s) \end{aligned} \tag{5.10}$$

with

$$h(z, s) = \begin{cases} 0 & s \in (-\infty, 0] \\ C(s/2\gamma_1(z)) & s \in (0, \infty). \end{cases}$$

If $\mathcal{H}$ is smooth relative to h it evolves essentially like a diffusion.

5.2.1. Low-frequency limit. A simple characterization of $\mathcal{H}$ can be obtained in the low-frequency limit. Then $\bar{l}(z, \omega) \approx l$ over the support of $\hat{G}_f$. We therefore have

$$p(z, \tau_1 + \varepsilon\chi_\varepsilon + \varepsilon^2 s) \approx [G_f \star \mathcal{N}](s) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{5.11}$$

where $\mathcal{H}$ is approximated by $\mathcal{N}$, a centred Gaussian pulse with variance $V = V(z) = (l/2)\int_0^z \gamma_1^2(s)\, \mathrm{d}s$.

We compare this approximation with the corresponding one in the strongly heterogeneous case given in (5.4). First, in both cases the deterministic change in the pulse shape is determined by convolution, on the scale of the probing pulse, with a Gaussian of squared width $(l/2)\int_0^z \gamma_1^2(s)\, \mathrm{d}s$. Second, the random correction to the travel time is defined analogously in the two cases. By the central limit theorem the scaled travel time correction, $\chi_\varepsilon/\varepsilon$, is approximately a Gaussian random variable with variance $V(z) = (l/2)\int_0^z \gamma_1^2(s)\, \mathrm{d}s$.

5.2.2. Connection to a random sum. Above we gave a simple interpretation of the pulse-shaping function $\mathcal{H}$ in the low-frequency limit. We next present a new interpretation of the approximation in the general case. To obtain a more transparent expression for $\mathcal{H}$ we write it as

$$\begin{aligned}\mathcal{H}(z,s) &= 1/(2\pi)\int \exp\left[-\omega^2\int_0^z \tfrac{1}{4}\gamma_1^2(u)\,\bar{l}(u,\omega)\,\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\omega s}\right]\mathrm{d}\omega \\ &= 1/(2\pi)\int \exp\left[-\tfrac{1}{8}\mathrm{i}\omega C(0)\int_0^z \gamma_1(u)\,\mathrm{d}u + az\left(-1+\int_0^\infty f(z,u)\,\exp(\mathrm{i}\omega u)\,\mathrm{d}u\right)\right] \\ &\quad\times \mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}\omega \end{aligned}\tag{5.12}$$

where the second equation was obtained by using integration by parts. We also use the notation $a \equiv -C'(0^+)/16$

$$f(z,u) \equiv \begin{cases} 0 & u \in (-\infty, 0] \\ -(C'(0^+)z)^{-1}\displaystyle\int_0^z C''\big(\tfrac{1}{2}u\,\gamma_1(s)\big)/(2\gamma_1(s))\,\mathrm{d}s & u \in (0,\infty). \end{cases}$$

We assume that $C'(0^+) < 0$, which is the case for a rough medium. Note that $\int_0^\infty f(z,u)\,\mathrm{d}u = 1$, and that if $\gamma_1(z) \equiv \gamma_1$ the function $f(z,u)$ is just a scaled version of the second derivative of the covariance function of ν.

Using the law of the iterated logarithm [17] it follows that in the weakly heterogeneous case

$$\tau(z) = \tau_1 + \varepsilon\chi_\varepsilon - (\varepsilon^2 C(0)/8)\int_0^z \gamma_1(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon^3\sqrt{\log\log\varepsilon^{-1}}) \tag{5.13}$$

where

$$\tau(z) = \int_{z_s}^z \gamma_1(s)\sqrt{1+\varepsilon\nu(s/\varepsilon^2)}\,\mathrm{d}s$$

is the first arrival time at depth z. Thus, in view of (5.12), we find that

$$p(z, \tau(z)+\varepsilon^2 s) \sim [G_f \star \tilde{H}](s) \qquad \text{as} \quad \varepsilon \downarrow 0$$

with $\tilde{\mathcal{H}}$ given by

$$\begin{aligned}\tilde{\mathcal{H}}(z,s) &= 1/(2\pi)\int \exp\left[az\left(-1+\int_0^\infty f(z,u)\,\exp(\mathrm{i}\omega u)\,\mathrm{d}u\right)\right]\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}\omega \\ &= p_0(az)\,\delta(s) + \sum_{n=1}^\infty p_n(az) f^{n*}(z,s).\end{aligned}$$

Here, $p_n(\lambda)$ is the discrete Poisson distribution with parameter $\lambda = az$

$$p_n(\lambda) = \mathrm{e}^{-\lambda}(\lambda)^n/n!.$$

This shows that, since $f(z,u) = 0$ for $s < 0$, we have obtained a strictly causal approximation. If $f \geqslant 0$, which is the case if ν is exponentially correlated, we obtain a characterization of $\tilde{\mathcal{H}}$ as the distribution of a random sum. Then $\tilde{\mathcal{H}}$ approaches the Gaussian distribution as $z \uparrow \infty$ by a generalization of the central limit theorem ([17], p 265). The width and centring of $\tilde{\mathcal{H}}$ are defined in terms of the first and second moments of f, which are

$$m_1(z) = \int_0^\infty uf(z,u)\,\mathrm{d}u = -2C(0)/(zC'(0^+))\int_0^z \gamma_1(s)\,\mathrm{d}s$$

$$m_2(z) = \int_0^\infty u^2 f(z,u)\,\mathrm{d}u = -8l/(zC'(0^+))\int_0^z \gamma_1^2(s)\,\mathrm{d}s.$$

Making use of the formulae for the moments of a random sum we obtain the delay, μ, and the squared width, σ^2, of $\tilde{\mathcal{H}}$ as

$$\mu = az\, m_1(z) = \tfrac{1}{8}C(0)\int_0^z \gamma_1(s)\,\mathrm{d}s$$

$$\sigma^2 = az\,(m_2(z) - m_1^2(z)) + az\, m_1^2(z)$$

$$= az\, m_2(z) = \tfrac{1}{2}l\int_0^z \gamma_1^2(s)\,\mathrm{d}s.$$

In view of (5.13) the resulting approximation for $\tilde{\mathcal{H}}$ conforms with the low-frequency approximation for $\mathcal{H}$ given in (5.11). Thus, the low-frequency approximation is also valid in the limit when the depth becomes large.

Consider now the special case when $\gamma_1(z) \equiv \gamma_1$ and when the fluctuations are exponentially correlated, that is, the covariance of the fluctuations is given by $E[\nu(0)\,\nu(s)] = C(0)\,\mathrm{e}^{-s/r}$. Then we obtain $\tilde{\mathcal{H}}$ explicitly as

$$\tilde{\mathcal{H}}(\mathcal{Z},\mathcal{T}) = \mathrm{e}^{-\mathcal{Z}}[\delta(\mathcal{T}) + \mathrm{e}^{-\mathcal{T}}\, d\sqrt{\mathcal{Z}/\mathcal{T}}\, I_1(2\sqrt{\mathcal{Z}\mathcal{T}})] \tag{5.14}$$

with $\mathcal{Z} = z[C(0)/(16r)]$, $\mathcal{T} = s/(2\gamma_1 r) \equiv sd$ and I_1 being the modified Bessel function of order 1. In figure 3 we show $\tilde{\mathcal{H}}$ for different relative propagation distances, that is, for $\sqrt{\mathcal{Z}} \in \{1.5, 2, 4, 6\}$. As the wave penetrates deeper into the medium we see that the associated pulse-shaping function loses its impulsive character and approaches a Gaussian pulse. The delta function part of $\tilde{\mathcal{H}}$ is indicated by the stars. These correspond to the part of a pulse that has 'tunnelled' undistorted through the medium. This part decays with the travelling length.

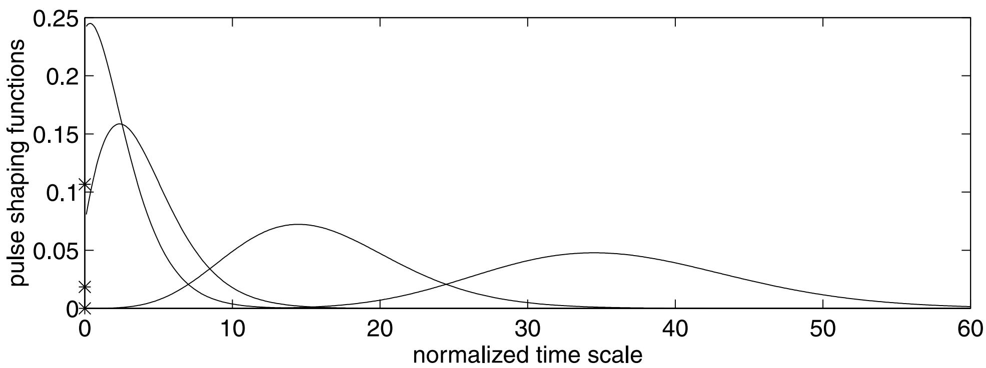

Figure 3. The pulse-shaping function $\tilde{\mathcal{H}}$ plotted as a function of normalized time $\mathcal{T}$ for a set of different relative travel lengths $\mathcal{Z}$. As the travel length becomes larger the pulse-shaping function becomes broader and approaches the Gaussian pulse shape, this corresponds to more broadening of the propagating pulse as it reaches deeper into the medium. The stars in the figure corresponds to the part of the pulse that has propagated undistorted through the medium, the delta function part of $\tilde{\mathcal{H}}$.

5.3. The pulse in the effective medium frame of reference

We show how the *mean* of the transmitted pulse, the coherent pulse, behaves when we observe it in the deterministic time frame defined by the effective medium parameters. In the derivation we follow [10, 12]. In the case of a weakly heterogeneous medium we consider the large-depth or low-frequency regimes. It then follows from the above results that the transmitted pulse is

$$p_k(z, \tau_1 + \varepsilon^k(\chi_\varepsilon/\varepsilon + s)) \sim [G_f \star \mathcal{N}_{V(z)}](s)$$
$$= (1/2\pi) \int \hat{G}_f(z, \omega)\, \mathrm{e}^{-\omega^2 V(z)/2}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \qquad (5.15)$$

with $V(z) = (l/2)\int_0^z \gamma_1(s)^2\, \mathrm{d}s$ and where $k = 1$ or 2 for strongly (5.1) or weakly (5.8) heterogeneous media, respectively. As above $G_f(z, (t - \gamma_1)/\varepsilon^k)$ is the transmitted pulse in the effective or deterministic medium and τ_1 is given by (5.3). We use the notation $\mathcal{N}_V$ for a Gaussian pulse shape with variance V. The pulse in the effective medium time frame is

$$p_k(z, \tau_1 + \varepsilon^k s) \sim (1/2\pi) \int \hat{G}_f(z, \omega)\, \mathrm{e}^{-\omega^2 V(z)/2}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{e}^{\mathrm{i}\omega\chi_\varepsilon(z)/\varepsilon}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0. \qquad (5.16)$$

Note that

$$\chi_\varepsilon(z)/\varepsilon = \int_0^z \gamma_1(s)\, \nu(s/\varepsilon^2)/(2\varepsilon)\, \mathrm{d}s$$

is, for small ε, approximately a Gaussian *random* variable with variance $V(z)$. The mean of the pulse in this frame can be obtained by integrating (5.16) with respect to the density for $\chi_\varepsilon(z)/\varepsilon$. We then find that

$$E[p_k(z, \tau_1 + \varepsilon^k s)] \sim (1/2\pi) \iint \hat{G}_f(z, \omega)\, \mathrm{e}^{-\omega^2 V(z)/2}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{e}^{\mathrm{i}\omega x}\mathrm{e}^{-x^2/2V(z)}/\sqrt{2\pi V(z)}\, \mathrm{d}x\, \mathrm{d}\omega$$
$$= (1/2\pi) \iint \hat{G}_f(z, \omega)\, \mathrm{e}^{-\omega^2 V(z)/2}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{e}^{-\omega^2 V(z)/2}\, \mathrm{d}x\, \mathrm{d}\omega$$
$$= (1/2\pi) \iint \hat{G}_f(z, \omega)\, \mathrm{e}^{-\omega^2 2V(z)/2}\, \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}x\, \mathrm{d}\omega$$
$$= [G_f \star \mathcal{N}_{2V(z)}](s) \qquad \text{as} \quad \varepsilon \downarrow 0.$$

Thus, the spreading of the coherent pulse in this time frame is as in (5.15) but with *twice* the variance for the pulse-shaping function.

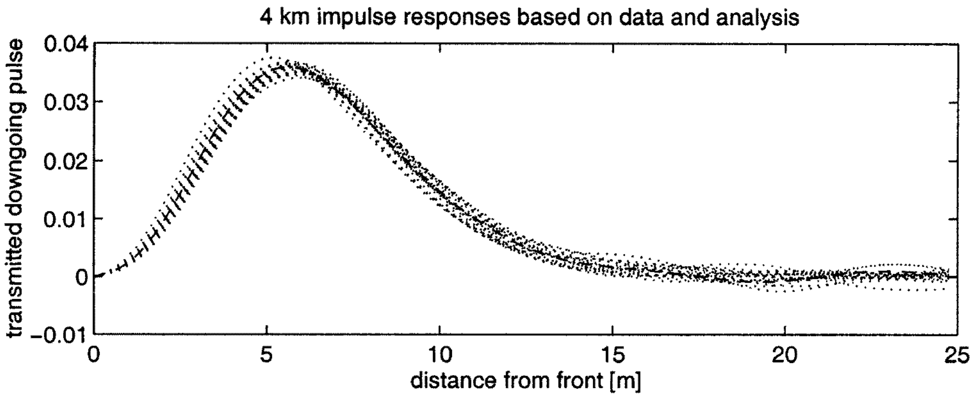

Figure 4. The transmitted pulse shapes obtained by propagating an impulse through realizations of a synthetic medium. The 20 dotted curves correspond to different realizations of the medium, all of length 4 km. They agree well with the O'Doherty–Anstey approximation shown by the broken curve. Note that all pulses are plotted relative to their first arrival time. We used a random medium model corresponding to $\varepsilon \approx 0.1$ and a 'correlation length' $\varepsilon^2 l \approx 0.5$ m.

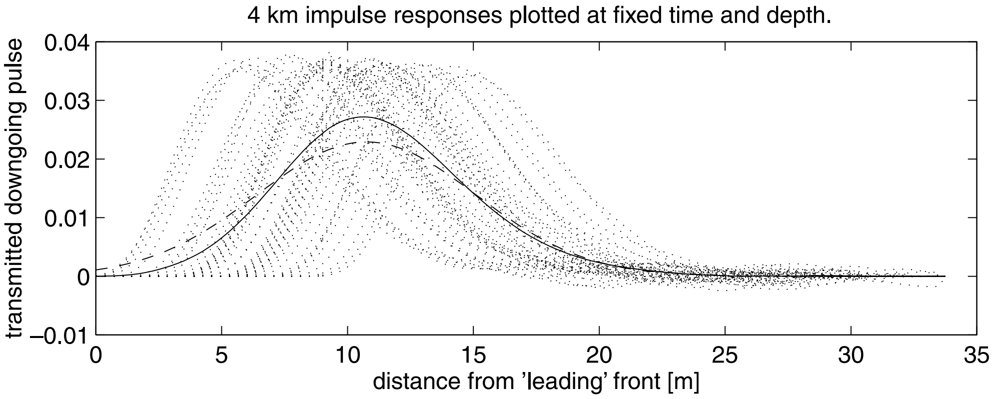

Figure 5. The dotted curves show transmitted pulses associated with 20 different realizations of the medium. The pulses are plotted relative to a fixed time frame, in which case they do not stabilize. The full curve is the average of the transmitted traces and the broken curve a Gaussian with variance twice that of the individual transmitted pulses. We used a random medium model corresponding to $\varepsilon \approx 0.1$ and a 'correlation length' $\varepsilon^2 l \approx 0.5$ m.

Next we present a numerical example that illustrates the above results. Based on a random medium model whose parameters are estimated from North Sea well log data [37] we simulated a set of realizations and propagated a pulse through them. The medium is weakly heterogeneous. In figure 4 we plot the transmitted pulses relative to the random frame when the pulse at the surface was close to an impulse. Note the stabilization phenomenon and convergence to a Gaussian pulse shape. In figure 5 we plot the pulses relative to a *fixed* time frame, so they disperse because of the random travel time component. The full curve shows the mean or coherent pulse and the broken curve a Gaussian pulse with the theoretical variance.

6. Pulse shaping for locally layered media

In this section we consider three-dimensional wave propagation and present the pulse-shaping approximation for the locally layered medium defined in section 3. This approximation is our main new result and it can be interpreted as a combination of the high-frequency approximation of section 4 with the layered pulse-shaping results of the previous section.

6.1. Locally layered strongly heterogeneous media.

Let $\boldsymbol{u}$, p solve (3.1)–(3.3) and (3.6) with $\Phi(\boldsymbol{x}, z) = z$. Furthermore, let τ_1 solve the eiconal equation (4.11) associated with $\gamma_1(\boldsymbol{x}, z)$ and a point source at $(\mathbf{0}, z_s)$. Assume that the characteristic ray segment Γ between the source and the observation point $(\boldsymbol{x}, z)$ is nowhere horizontal (see figure 2). Furthermore, assume that τ_1 is smooth in a neighbourhood of this path. Then for $z > 0$, with probability one

$$p(\boldsymbol{x}, z, \tau_1 + \chi_\varepsilon + \varepsilon s) \sim [G_f \star \mathcal{N}](s) = \int G_f(\boldsymbol{x}, z, s - \acute{s})\, \mathcal{N}(\boldsymbol{x}, z, \acute{s})\, \mathrm{d}\acute{s}$$

$$= (1/2\pi) \int \hat{G}_f(\boldsymbol{x}, z, \omega) \times \exp\left[-\omega^2 \int_{\Gamma^*} (\gamma_1^2\, l/4\cos(\theta))\, \mathrm{d}u\right] \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{6.1}$$

where $G_f(\boldsymbol{x}, z, (t - \tau_1)/\varepsilon)$ is the transmitted pressure pulse in the deterministic case when $\nu \equiv 0$. The ray segment Γ^* is the part of Γ that goes between the surface $z = 0$ and the point of observation $(\boldsymbol{x}, z)$. The high-frequency approximation for G_f is given in (4.14). The pulse-shaping function $\mathcal{N}$ is a centred Gaussian pulse with variance V and

$$\begin{aligned} V &= (l/2) \int_{\Gamma^*} \gamma_1^2 \cos(\theta)^{-1}\, \mathrm{d}u \\ \tau_1 &= \int_{\Gamma} \gamma_1\, \mathrm{d}u \\ \chi_\varepsilon &= \int_{\Gamma^*} \tfrac{1}{2}\gamma_1 \nu\, \mathrm{d}u \\ l &\equiv \int_0^\infty E[\nu(0)\, \nu(u)]\, \mathrm{d}u \\ \gamma_1 &= \sqrt{\rho_0/K_1} \end{aligned} \tag{6.2}$$

where u is the arc length along the path Γ and $\cos(\theta(\boldsymbol{x}, z)) = \tau_{1,z}/|\nabla\tau_1|$, thus, θ is the angle between Γ and the vertical direction.

The pulse-shaping function $\mathcal{N} = \mathcal{N}(u, s)$ solves

$$\mathcal{N}_u = D\,\mathcal{N}_{ss} \qquad \mathcal{N}|_{u_0} = \delta(s).$$

In this diffusion equation u is the arc-length parameter of the characteristic ray segment between the source and observation point and plays the role of time, u_0 corresponds to $z = 0$. The 'diffusion coefficient' is

$$D(\boldsymbol{x}, z) = l\,\gamma_1^2(\boldsymbol{x}, z)/(4\cos(\theta(\boldsymbol{x}, z))).$$

The pulse-shaping function $\mathcal{N}$ is a probability distribution, hence the L_1 norm of a positive pulse is preserved by the convolution. The approximation (6.1) generalizes the classical O'Doherty–Anstey approximation of the one-dimensional case. It differs in that the random travel time correction χ_ε and the square width of the modulating pulse, V, are defined as integrals over the geometrical optics path Γ, which in the purely layered normally incident plane-wave case specializes to $(\mathbf{0}, (0, z))$. As in the layered case the approximation (6.1) modifies the usual high-frequency approximation for the effective medium in two important ways.

First, the arrival time of the transmitted pulse, defined as the centre of the impulse response, is random and is given by

$$\tau_\varepsilon = \tau_1 + \chi_\varepsilon = \int_\Gamma \tfrac{1}{2}\gamma_1(u)(1 + \nu(z(u)/\varepsilon^2))\, \mathrm{d}u$$

with Γ the characteristic path from the *source* to the point of observation. The travel time along Γ is given by

$$\tau = \int_\Gamma \gamma_1(u)\sqrt{(1 + \nu(z(u)/\varepsilon^2))}\, \mathrm{d}u$$

hence $\tau < \tau_\varepsilon$. Moreover, by the central limit theorem

$$\varepsilon^{-1}[\tau_\varepsilon - \tau_1] = \varepsilon^{-1} \int_\Gamma \tfrac{1}{2}\gamma_1(u)\, \nu(z(u)/\varepsilon^2)\, \mathrm{d}u \to X \qquad \text{as} \quad \varepsilon \downarrow 0$$

with X a centred Gaussian random variable with variance V defined as in (6.2). Therefore, we see that the discrepancy between the centre of the impulse response and the effective medium arrival time is a mean-zero $\mathcal{O}(\varepsilon)$ random quantity and is hence on the scale of the probing pulse.

Second, the scattering associated with the fluctuations causes a smearing of the travelling pulse. The asymptotic characterization of this phenomenon is through convolution with the Gaussian pulse $\mathcal{N}$. The convolution is on the scale of the probing pulse, and hence interacts strongly with it. The width of the Gaussian is given in terms of moments of medium parameters along the path Γ *only*, and does not depend on the particular realization. The pulse shaping, though only visible after a long distance, is a *local* phenomenon. The random modulation of the medium parameters, on the finest scale of the model, causes energy to be scattered over to the up-propagating wave mode, but this energy is quickly scattered back again due to the fluctuations. Hence, only a small amount of energy is carried by the up-propagating wave mode but it is important because the continuous random channelling of energy gradually delays the pulse relative to the first arrival and causes its shape to diffuse and approach a Gaussian. If there is a lot of structure in the fluctuations, that is, strong correlations, then l will be relatively large. Coherence and strong variability in the random medium modulation means that the random scattering is associated with a stronger smearing of the pulse. Observe that the 'effective' correlations length is $l/\cos(\theta)$. If the pulse propagates with a shallow angle relative to the layering it 'sees' a medium with stronger coherence.

It follows from the above that the analysis presented in section 5.3 prevails in the locally layered case. That is, if we observe the pulse in a deterministic effective medium time frame we see the *mean* or coherent pulse that is the deterministic pulse convolved with a centred Gaussian with variance $2V$, to leading order.

Note also that since $\mathcal{N}$ is strictly positive for all arguments, the approximation (6.1), being a diffusive transport approximation, actually violates causality. However, this concerns only the tail which is exponentially small.

Figure 6 illustrates the above results. The paths Γ_i are geometric optics ray paths associated with a point source at $(\mathbf{0}, z_s)$ and u_i the arc-length parameters along these. For a set of travel times we plot the pulse in the deterministic or effective medium with dotted curves. To leading order the evolution of the pulse shape along the rays is described by the first-order transport equations of the geometric optics approximation. The full curves illustrate the corresponding pulse shapes in the locally layered medium. They are blurred somewhat because of scattering associated with the microscale medium fluctuations. Moreover, the travel time along the ray is corrected by a mean zero random variable. The diffusion depends only on the statistics of the medium fluctuations, whereas the travel time correction depends on the detailed structure of the medium fluctuations along the geometric optics ray path.

6.2. Simplification for purely layered media

In the purely layered case, $\gamma_1 = \gamma_1(z)$, the above simplifies because the phase and ray paths can be written more explicitly. For the approximation (6.1) we obtain the following expressions

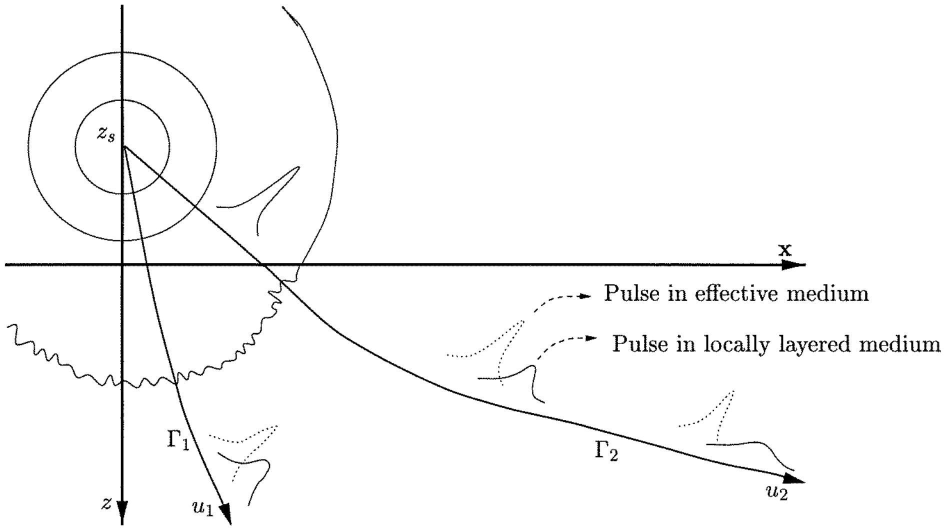

Figure 6. Pulse shaping in a locally layered medium. We plot the pulse shapes as it evolves along two different geometrical optics ray paths. The dotted curves correspond to the pulse shape in the effective medium. The pulse shapes in the locally layered medium are given by the full curves. The blurring or smearing of the pulse shapes is due to microscale scattering and is a *deterministic* effect. The travel time of the pulse is corrected by a mean zero *random* variable.

for the quantities involved:

$$\chi_\varepsilon = \int_0^z \tfrac{1}{2}\gamma_1 \nu \cos(\theta)^{-1}\,\mathrm{d}s$$

$$V = (l/2)\int_0^z \gamma_1^2 \cos(\theta)^{-2}\,\mathrm{d}s$$

where s is the depth variable. The angle θ is defined by

$$\cos(\theta(z)) = \sqrt{1 - \kappa^2/\gamma_1^2(z)}$$

with κ^2 determined by

$$\int_{z_s}^{z} [\gamma_1(s)^2/\kappa^2 - 1]^{-1/2}\,\mathrm{d}s = \|\boldsymbol{x}\|_2$$

where $(\boldsymbol{x}, z)$ is the point of observation. The pulse-shaping approximation associated with a point source was first discussed in [9]. There the approximation is presented in the context of a weakly heterogeneous discretely layered medium. A rigorous derivation of the approximation in the case of a point source over a strongly heterogeneous medium is given in [13] using Ito calculus. In appendix A we present an alternative derivation based on limit theorems for the moments of certain stochastic ordinary differential equations.

6.3. The homogeneous case

In the special case of a uniform background medium, that is $\gamma_1(\boldsymbol{x}, z) = \gamma_1$, we find using (4.15) the approximation

$$
\begin{aligned}
p(\boldsymbol{x}, z, \gamma_1 r + \chi_\varepsilon + \varepsilon s) &\sim [G_f \star \mathcal{N}](s) \\
&\sim \frac{\gamma_1 \cos(\theta)}{4\pi r} [f' \star \mathcal{N}](s) \qquad \text{as} \quad \varepsilon \downarrow 0
\end{aligned} \tag{6.3}
$$

with

$$
\begin{aligned}
\chi_\varepsilon &= (\gamma_1/2) \int_0^z \nu \, \mathrm{d}s \cos(\theta)^{-1} \\
V &= (r\, \gamma_1^2\, l/2) \cos(\theta)^{-1} \\
\cos(\theta) &= (z - z_s)/r
\end{aligned}
$$

and r the distance from the source to the point of observation. Again, we see that the random layering is felt more strongly when the wave propagates with a shallow angle θ relative to the layering.

If we consider the pulse just beneath the source, $\theta = 0$, and make the change of variables $\mathcal{Z} = z/(\varepsilon^2 l)$ and $\mathcal{T} = s/(\gamma_1 \varepsilon l)$ we find that $\mathcal{N}$ solves

$$
\begin{aligned}
\mathcal{N}_{\mathcal{Z}} &= \tfrac{1}{4} \mathcal{N}_{\mathcal{T}\mathcal{T}} \\
\mathcal{N}(0, \mathcal{T}) &= \delta(\mathcal{T})
\end{aligned} \tag{6.4}
$$

and in terms of these variables $\mathcal{N}$, as a function of $\mathcal{T}$, is a Gaussian pulse of squared width $\mathcal{Z}/2$. Thus, if we refer to the correlation length of the fluctuations on the 'macroscopic' spatial scale, $\varepsilon^2 l$, as 'the correlation length', we conclude that when the pulse has reached N correlation lengths into the medium, the spatial support of the convolving pulse is about $2\sqrt{N}$ correlation lengths. Here we define spatial support as the interval containing about 99% of the pulse energy.

6.4. The weakly heterogeneous case.

Consider next the weakly heterogeneous case with source and medium parameters defined by (3.4) and (3.5), but with otherwise the same assumptions as in section 6.1. As above we assume $\Phi(\boldsymbol{x}, z) = z$. Then with probability one

$$
\begin{aligned}
p(\boldsymbol{x}, z, \tau_1 + \varepsilon \chi_\varepsilon + \varepsilon^2 s) &\sim [G_f \star \mathcal{H}](s) = \int G_f(\boldsymbol{x}, z, s - \acute{s})\, \mathcal{H}(\boldsymbol{x}, z, \acute{s})\, \mathrm{d}\acute{s} \\
&= (1/2\pi) \int \hat{G}_f(\boldsymbol{x}, z, \omega) \exp\left[-\omega^2 \int_{\Gamma^*} \left(\gamma_1^2\, \bar{l}/(4\cos(\theta)\right) \mathrm{d}u\right] \mathrm{e}^{-\mathrm{i}\omega s}\, \mathrm{d}\omega \\
&\quad \text{as} \quad \varepsilon \downarrow 0
\end{aligned} \tag{6.5}
$$

where

$$
\begin{aligned}
\bar{l} = \bar{l}(\boldsymbol{x}, z, \omega) &= \int_0^\infty C(s)\, \mathrm{e}^{\mathrm{i}\omega 2\gamma_1(\boldsymbol{x}, z) s}\, \mathrm{d}s \\
C(s) &= E[\nu(0)\, \nu(s)]
\end{aligned}
$$

with the other quantities being defined as in (6.1). Again, the transmitted pulse is defined analogously to the transmitted pulse in the one-dimensional case (5.9). It differs in that the paths in the integrals defining χ_ε and V have been generalized. In this scaling the support of the pulse is on the scale of the fluctuations and the pulse interacts with the statistical structure of the fluctuations, not only their intensity. Thus, now the pulse-shaping function depends on the whole autocovariance function of the fluctuations ν.

In the low-frequency limit, when the pulse becomes broad relative to the medium fluctuations and $\hat{G}_f(\boldsymbol{x}, z, \cdot)$ is narrowly supported at the origin, we obtain the approximation

$$p(\boldsymbol{x}, z, \tau_1 + \varepsilon\chi_\varepsilon + \varepsilon^2 s) \approx [G_f \star \mathcal{N}](s) \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{6.6}$$

The variance of the Gaussian pulse $\mathcal{N}$ is

$$V = (l/2) \int_{\Gamma^*} \gamma_1^2 \cos(\theta)^{-1} \, \mathrm{d}u.$$

Hence, the convolving pulse is defined as in the strongly heterogeneous case (6.1). The arrival time is approximated by $\tau_\varepsilon = \tau_1 + \varepsilon\chi_\varepsilon$. Thus, the approximations (6.1) and (6.6) differ only in their scaling. When time is scaled by ε^{-1} and ε^{-2} in the strong and weak noise cases, respectively, we can replace the question mark in figure 1, modulo the effective medium response, by a Gaussian pulse of squared width V, time-shifted by $\varepsilon^{-1}\chi_\varepsilon$.

The approximation (6.6) is also valid in the limit when the depth z becomes large. This follows by a central limit theorem argument applied to the following reformulation of (6.5):

$$p(\boldsymbol{x}, z, \tau + \varepsilon^2 s) \sim [G_f \star \tilde{\mathcal{H}}](s) \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{6.7}$$

The derivation parallels the one presented in section 5.2.2. In this formulation we centre with respect to τ, the integral of the local speed of sound along the characteristic ray from the source to the point of observation

$$\tau = \int_{\Gamma} \gamma_1 \sqrt{1 + \varepsilon\nu} \, \mathrm{d}u.$$

Moreover, we have

$$\tilde{\mathcal{H}}(\boldsymbol{x}, z, s) = p_0(\boldsymbol{x}, z)\,\delta(s) + \sum_{n=1}^{\infty} p_n(\boldsymbol{x}, z) F^{n*}(\boldsymbol{x}, z, s)$$

with

$$\begin{aligned} p_n(\boldsymbol{x}, z) &= \mathrm{e}^{-\lambda}\lambda^n/n! \\ \lambda &= a \cdot b \\ a &\equiv -C'(0^+)/16 \\ b &= \int_{\Gamma^*} \cos(\theta)^{-1} \, \mathrm{d}u. \end{aligned}$$

As before θ and γ_1 are evaluated along the path of integration Γ, the characteristic line segment between the source and the observation point. We assume that $C'(0^+) < 0$. The function F is defined by

$$F(\boldsymbol{x}, z, s) \equiv \begin{cases} 0 & s \in (-\infty, 0] \\ -(C'(0^+)b)^{-1} \displaystyle\int_{\Gamma^*} C''(s/(2\gamma_1)/(2\gamma_1 \cos(\theta)) \, \mathrm{d}u & s \in (0, \infty). \end{cases} \tag{6.8}$$

The pulse-shaping function $\tilde{\mathcal{H}}$ can be interpreted as the distribution of a random sum with p_n the discrete Poisson distribution with parameter λ. In section 5.2.2 we discuss a particular example for the pulse-shaping function $\tilde{\mathcal{H}}$, the one associated with a Markovian fluctuation process ν.

7. Derivation of the pulse-shaping approximation

In the next sections we derive the pulse-shaping approximations described above. We start out with a review of the one-dimensional case in sections 7.1 and 7.2. These results were first obtained in [1, 10, 14]. This review sets the stage for an analysis in the case of a locally layered medium. The derivation is formal, a rigorous derivation of the layered case follows from the analysis of the moment equations presented in appendix A. A corresponding analysis of moment equations for functionals that pertain to the locally layered case is given in [36].

7.1. Plane-wave pulses in strongly heterogeneous layered media

First, we present a derivation of the pulse-shaping approximation in the purely one-dimensional strongly heterogeneous case discussed in section 5.1. In this case a normally incident plane-wave pulse is impinging normally upon the strictly layered half-space $z > 0$. The governing equations are

$$\begin{aligned} &\rho u_t + p_z = F_\varepsilon(z,t) \\ &K_\varepsilon^{-1}(z)\, p_t + u_z = 0 \end{aligned} \tag{7.1}$$

with

$$\begin{aligned} &\rho(z) \equiv \rho_0 \\ &K_\varepsilon^{-1}(z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(z)(1+\nu(z/\varepsilon^2)) & z \in (0,\infty) \end{cases} \\ &F_\varepsilon(z,t) = f(t/\varepsilon)\,\delta(z - z_s). \end{aligned}$$

We seek an asymptotic approximation for the transmitted pressure. To this effect we convert (7.1) into a stochastic integro-differential equation for the time harmonic amplitudes of the down-going wave part of the travelling pulse. The approximation follows from this representation.

7.1.1. Decomposition in terms of up- and down-travelling waves. Define the scaled Fourier transform by

$$\hat{p}(z,\omega) = \int p(z,s)\,\mathrm{e}^{\mathrm{i}\omega s/\varepsilon}\,\mathrm{d}s$$

$$\hat{u}(z,\omega) = \int u(z,s)\,\mathrm{e}^{\mathrm{i}\omega s/\varepsilon}\,\mathrm{d}s.$$

From (7.1) we find for $z \geqslant 0$

$$\hat{p}_{zz} + [1+\nu(z/\varepsilon^2)](\omega/\varepsilon)^2\,\gamma_1^2(z)\,\hat{p} = 0 \tag{7.2}$$

with the slowness being defined by

$$\gamma_1(z) = \sqrt{\rho_0/K_1(z)}. \tag{7.3}$$

We parametrize $\hat{p}$ in terms of up- and down-travelling wave components and make the ansatz

$$\hat{p} = A\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon} \tag{7.4}$$

$$0 = A_z\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B_z\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon} \tag{7.5}$$

where τ_1 is defined in (4.5) and where $A = A(z,\omega)$ and $B = B(z,\omega)$ correspond, respectively, to the transmitted (positive z-direction) and reflected field components.

Upon a change of variables it can be shown that the ansatz formulated in (7.4) and (7.5) corresponds to (2.26) in [1].

7.1.2. Equations for the amplitudes. We now proceed to obtain equations governing the evolution of the amplitudes with respect to depth. Substituting (7.4) in (7.2) we obtain

$$[2\gamma_1 A_z + \gamma_{1,z} A - \mathrm{i}(\varepsilon/\omega) A_{zz}]\, \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} - [2\gamma_1 B_z + \gamma_{1,z} B + \mathrm{i}(\varepsilon/\omega) B_{zz}]\, \mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon}$$
$$= \mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu[A\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon}]$$

moreover, from (7.5) it follows that

$$[\gamma_1 A_z \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} - \gamma_1 B_z \mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon}] - \mathrm{i}(\varepsilon/\omega)[A_{zz}\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B_{zz}\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon}] = 0.$$

Combining the above two relations we find

$$[\gamma_1 A_z + \gamma_{1,z} A]\, \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} - [\gamma_1 B_z + \gamma_{1,z} B]\, \mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon} = \mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu[A\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon}]. \tag{7.6}$$

Finally, adding/subtracting a multiple γ_1 of (7.5) to/from (7.6) gives

$$\begin{aligned} 2\gamma_1 A_z + \gamma_{1,z} A &= \mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu[A + B\mathrm{e}^{-2\mathrm{i}\omega\tau_1/\varepsilon}] + \gamma_{1,z} B\mathrm{e}^{-2\mathrm{i}\omega\tau_1/\varepsilon} \\ 2\gamma_1 B_z + \gamma_{1,z} B &= -\mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu[A\mathrm{e}^{2\mathrm{i}\omega\tau_1/\varepsilon} + B] + \gamma_{1,z} A\mathrm{e}^{2\mathrm{i}\omega\tau_1/\varepsilon}. \end{aligned} \tag{7.7}$$

We return to the above calculations in a more general framework in section 7.4. Consider next the transformation

$$\begin{aligned} \alpha &= A\, \exp\left[\int_0^z [\mathrm{d}[\ln[\sqrt{\gamma_1}]]/\mathrm{d}s - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\, \mathrm{d}s\right] \\ \beta &= B\, \exp\left[\int_0^z [\mathrm{d}[\ln[\sqrt{\gamma_1}]]/\mathrm{d}s + \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\, \mathrm{d}s\right]. \end{aligned} \tag{7.8}$$

This transformation corresponds to compensating for a random travel time correction caused by the random medium fluctuations and also for the transformation of the pulse shape that is due to variations in the deterministic background medium. We arrive at the equations

$$\mathrm{d}\alpha/\mathrm{d}z = \zeta\beta \qquad \mathrm{d}\beta/\mathrm{d}z = \overline{\zeta}\alpha \tag{7.9}$$

with

$$\zeta(z,\omega) = \left[\frac{\mathrm{i}\omega\gamma_1(z)\,\nu(z/\varepsilon^2)}{2\varepsilon} + \frac{\mathrm{d}[\ln[\sqrt{\gamma_1(z)}]]}{\mathrm{d}z}\right]\mathrm{e}^{-2\mathrm{i}\omega\tau_\varepsilon/\varepsilon} \tag{7.10}$$

and

$$\tau_\varepsilon(z) = \int_{z_s}^{z} \gamma_1(s)(1 + \nu(s/\varepsilon^2)/2)\, \mathrm{d}s. \tag{7.11}$$

Note that α is 'centred' with respect to the frame τ_ε that is slightly different from the one moving with effective medium slowness γ_1. The *random* travel time centring, τ_ε, makes the system (7.9) purely 'off-diagonal'.

In the high-frequency approximation the coupling of the amplitudes in (7.9) that is due to the term '$\mathrm{d}[\ln[\sqrt{\gamma_1}]]/\mathrm{d}z$' in ζ can be ignored. In the effective medium approximation we also ignore the stochastic coupling between the amplitudes, that is, the coupling due to the term '$\mathrm{i}\omega\gamma_1\nu/(2\varepsilon)$'. As we show in the next section, this stochastic coupling causes a small modulation of the pulse which becomes appreciable for long propagation distances, on the scale on which the deterministic background medium varies.

We assume that the background medium variations are smooth so that the reflected pulse is small and obtain from (7.9) the expression

$$\beta(z,\omega) = -\int_z^\infty \overline{\zeta(s,\omega)}\,\alpha(s,\omega)\, \mathrm{d}s$$

for the up-going amplitude and the stochastic integro-differential equation

$$\frac{d\alpha}{dz}(z,\omega) = -\int_z^\infty \zeta(z,\omega)\,\overline{\zeta(s,\omega)}\,\alpha(s,\omega)\,ds \tag{7.12}$$

for the down-going amplitude. In the next section, we offer a simple heuristic argument showing how the solution of (7.12) can be characterized for small ε.

7.1.3. Stabilization of the pulse. Take the expected value of (7.12) to obtain

$$\frac{dE[\alpha(z,\omega)]}{dz} = -\int_z^\infty E[\zeta(z,\omega)\,\overline{\zeta(s,\omega)}\,\alpha(s,\omega)]\,ds.$$

The effective medium approximation suggests that the transformation of the travelling pulse, due to the random fluctuations of the medium parameters, occurs on a scale which is slow relative to that of the random fluctuations. The heuristic argument now rests on the following approximation motivated by this observation:

$$E[\zeta(z,\omega)\,\overline{\zeta(s,\omega)}\,\alpha(s,\omega)] \sim E[\zeta(z,\omega)\,\overline{\zeta(s,\omega)}]E[\alpha(s,\omega)] \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{7.13}$$

Furthermore, for $z > 0$ and $\overline{\alpha}$ smooth

$$\int_z^\infty E[\zeta(z,\omega)\,\overline{\zeta(s,\omega)}]\,\overline{\alpha}(s)\,ds \sim \omega^2 D(z)\,\overline{\alpha}(z) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{7.14}$$

with $D(z) = \gamma_1^2(z)l/4$. The correlation length l is defined in (6.2). Hence, assuming $E[\alpha(z,\omega)]$ to be smooth, we obtain

$$E[\alpha(z,\omega)] \sim a(z,\omega) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{7.15}$$

with a solving for $z > 0$

$$\frac{da}{dz}(z,\omega) = -\omega^2 D(z)\,a(z,\omega) \tag{7.16}$$

and $a(0,\omega) = A(0,\omega)$. We find from (7.4) and (7.8)

$$\begin{aligned} p(z,t) \sim p_{\mathrm{down}}(z,t) &\equiv (1/2\pi\varepsilon)\int A(z,\omega)\,e^{i\omega(\tau_1 - t)/\varepsilon}\,d\omega \\ &= \left\{\exp\left[-\int_0^z d[\ln[\sqrt{\gamma_1(s)}\,]\right]/(2\pi\varepsilon)\right\}\int \alpha(z,\omega)\,e^{i\omega(\tau_\varepsilon - t)/\varepsilon}\,d\omega \\ &= \left[\sqrt{\gamma_0/\gamma_1(z)}/(2\pi\varepsilon)\right]\int \alpha(z,\omega)\,e^{i\omega(\tau_\varepsilon - t)/\varepsilon}\,d\omega. \end{aligned} \tag{7.17}$$

Next, we derive an approximation for p by substituting α with a in (7.17). This leads to the O'Doherty–Anstey pulse-shaping approximation. Define

$$\bar{p}(z,t) = \left[\sqrt{\gamma_0/\gamma_1(z)}/(2\pi\varepsilon)\right]\int A(0,\omega)\,\exp\left[-\omega^2\int_0^z D(s)\,ds\right]e^{i\omega(\tau_\varepsilon - t)/\varepsilon}\,d\omega. \tag{7.18}$$

If we centre with respect to the random phase and use the time scaling of the source we obtain from (7.18)

$$\bar{p}(z,\tau_\varepsilon + \varepsilon s) = [f_0 \star \mathcal{N}](s) \qquad \text{as} \quad \varepsilon \downarrow 0$$

where τ_ε is defined in (7.11) and where f_0 is the high-frequency approximation associated with the effective medium parameters

$$f_0(z,s) = 2^{-1}\sqrt{\gamma_0/\gamma_1(z)}\,f(s). \tag{7.19}$$

The pulse-shaping function $\mathcal{N}$ is defined by

$$\mathcal{N}(z,s) = (1/2\pi)\int \exp\left[-\omega^2 \int_0^z D(v)\,\mathrm{d}v\right] \mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}\omega$$

hence, $\mathcal{N}$ is a Gaussian pulse of squared width $2\int_0^z D(u)\,\mathrm{d}u = (l/2)\int_0^z \gamma_1^2(u)\,\mathrm{d}u$. This is the O'Doherty–Anstey pulse-shaping approximation introduced in section 5.1 with $\mathcal{N}$ solving the diffusion equation (5.5).

A key aspect of the O'Doherty–Anstey theory is that in the above *random time frame* the transmitted pressure pulse is described asymptotically by a *deterministic* pulse. We show this next. Using (7.15), (7.17) and (7.18) we obtain the following expression for the variance:

$$E[(p_{\text{down}}(z,\tau_\varepsilon+\varepsilon s) - \bar{p}(z,\tau_\varepsilon+\varepsilon s))^2] \sim \gamma_0/(4\pi^2\gamma_1(z)\varepsilon^2)\iint \Big\{ E[\alpha(z,\omega_1)\,\alpha(z,\omega_2)] \\ -A(0,\omega_1)A(0,\omega_2)\exp\left[-(\omega_1^2+\omega_2^2)\int_0^z D(s)\,\mathrm{d}s\right]\Big\}\,\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)s}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \\ \text{as}\quad \varepsilon \downarrow 0. \tag{7.20}$$

Note that

$$\frac{\mathrm{d}E[\alpha(z,\omega_1)\,\alpha(z,\omega_2)]}{\mathrm{d}z} = -\int_z^\infty \{E[\zeta(z,\omega_1)\,\overline{\zeta(s,\omega_1)}\,\alpha(s,\omega_1)\,\alpha(z,\omega_2)] \\ +E[\zeta(z,\omega_2)\,\overline{\zeta(s,\omega_2)}\,\alpha(z,\omega_1)\,\alpha(s,\omega_2)]\}\,\mathrm{d}s.$$

Therefore, if we again make an assumption about 'locality' as in (7.13) we find

$$E[\alpha(z,\omega_1)\,\alpha(z,\omega_2)] \sim h(z,\omega_1,\omega_2) \qquad \text{as}\quad \varepsilon\downarrow 0$$

with h solving for $z > 0$

$$\frac{\mathrm{d}h}{\mathrm{d}z}(z,\omega_1,\omega_2) = -(\omega_1^2+\omega_2^2)D(z)h(z,\omega_1,\omega_2). \tag{7.21}$$

More precisely,

$$E[\alpha(z,\omega_1)\,\alpha(z,\omega_2)] = E[\alpha(z,\omega_1)]E[\alpha(z,\omega_2)] \\ = A(0,\omega_1)A(0,\omega_2)\exp\left[-(\omega_1^2+\omega_2^2)\int_0^z D(s)\,\mathrm{d}s\right].$$

Thus, from (7.20) we indeed find that $p_{\text{down}}(z,\tau_\varepsilon+\varepsilon s) \sim \bar{p}(z,\tau_\varepsilon+\varepsilon s)$.

This is the remarkable stabilization aspect of the O'Doherty–Anstey theory. The random fluctuations of the pulse, when observed in the appropriate *random* time frame are negligible for small ε. Note that α is complex and it does not follow from (7.21) that the time harmonic amplitude itself stabilizes. In fact, it does not. Therefore, the above result cannot be obtained by considering the evolution of the different harmonic amplitudes in isolation, which is done for instance in [35].

That the harmonic amplitudes evaluated at different frequencies are uncorrelated is the key property that gives stabilization. There is, in general, a random time correction such that the resulting harmonic amplitudes have this property that stabilizes the pulse. Actually, in the above problem the amplitudes are statistically independent when evaluated at different frequencies [24].

For general multidimensional problems, if there exists a phase shift, that is, a time correction such that the harmonic Green functions are uncorrelated to leading order when evaluated at frequencies that are distinct then we have stabilization. In section 7.4 we show that for a locally layered medium this phase shift is obtained as an integral of medium fluctuations along the geometrical optics ray from the source to the point where we observe the pulse. The ray is obtained from the deterministic effective medium parameters.

7.2. The weakly heterogeneous layered case

Next, we turn our attention to the weakly heterogeneous case. The argument leading to an approximation of the transmitted pulse in the weakly heterogeneous case is a modification of the one in the strongly heterogeneous case. The governing equations pertaining to the weakly heterogeneous case are as in (7.1). However, now the source and the medium parameters are given by

$$\rho(z) \equiv \rho_0$$

$$K_\varepsilon^{-1}(z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(z)(1 + \varepsilon\nu(z/\varepsilon^2)) & z \in (0, \infty) \end{cases}$$

$$F_\varepsilon(z, t) = f(t/\varepsilon^2)\, \delta(z - z_s).$$

In the weakly heterogeneous case we define the Fourier transform by

$$\hat{p}(z, \omega) = \int p(z, s)\, \mathrm{e}^{\mathrm{i}\omega s/\varepsilon^2}\, \mathrm{d}s$$

$$\hat{u}(z, \omega) = \int u(z, s)\, \mathrm{e}^{\mathrm{i}\omega s/\varepsilon^2}\, \mathrm{d}s.$$

The appropriate ansatz is now

$$\hat{p} = A\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon^2} + B\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon^2}$$

$$0 = A_z\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon^2} + B_z\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon^2}$$

where τ_1 is defined in (4.5). Proceeding as before we find upon the change of variables

$$\alpha = A\, \exp\left[\int_0^z [\mathrm{d}[\ln[\sqrt{\gamma_1}]/\mathrm{d}s - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\, \mathrm{d}s\right]$$

$$\beta = B\, \exp\left[\int_0^z [\mathrm{d}[\ln[\sqrt{\gamma_1}]/\mathrm{d}s + \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\, \mathrm{d}s\right]$$

that the amplitudes satisfy

$$\mathrm{d}\alpha/\mathrm{d}z = \zeta\beta \qquad \mathrm{d}\beta/\mathrm{d}z = \overline{\zeta}\alpha \tag{7.22}$$

with

$$\zeta(z, \omega) = \left[\frac{\mathrm{i}\omega\gamma_1(z)\, \nu(z/\varepsilon^2)}{2\varepsilon} + \frac{\mathrm{d}[\ln[\sqrt{\gamma_1(z)}]}{\mathrm{d}z}\right] \mathrm{e}^{-2\mathrm{i}\omega\tau_\varepsilon(z)/\varepsilon^2}. \tag{7.23}$$

It follows that α satisfies (7.12), as before, with ζ defined as above. Note that in this case we define

$$\tau_\varepsilon = \int_{z_s}^z \gamma_1(s)(1 + \varepsilon\nu(s/\varepsilon^2)/2)\, \mathrm{d}s. \tag{7.24}$$

In the weakly heterogeneous case we motivate the approximation (7.13) by the fact that the coupling between the propagating pulse and the random process ν is weak. Moreover, note that for $z > 0$ and $\overline{\alpha}$ smooth

$$\int_z^\infty E[\zeta(z, \omega)\, \overline{\zeta(s, \omega)}]\, \overline{\alpha}(s)\, \mathrm{d}s \sim \omega^2 D(z, \omega) \qquad \text{as} \quad \varepsilon \downarrow 0$$

with $D(z,\omega) = \gamma_1^2(z)\,\bar{l}(z,\omega)/4$, and where $\bar{l}$ is defined by

$$\bar{l}(z,\omega) \equiv \int_0^\infty E[\nu(0)\,\nu(s)]\,\mathrm{e}^{\mathrm{i}\omega 2\gamma_1(z)s}\,\mathrm{d}s$$
$$\equiv \int_0^\infty C(s)\,\mathrm{e}^{\mathrm{i}\omega 2\gamma_1(z)s}\,\mathrm{d}s.$$

Thus, the 'diffusion parameter' $D(z,\omega)$ depends on the spatial autocovariance function of the process ν and not only the correlation length as was the case above, which reflects the fact that the probing pulse now is defined on the same scale as the fluctuations. By an argument similar to that preceding (7.15) we find $E[\alpha(z,\omega)] \sim a(z,\omega)$ as $\varepsilon \downarrow 0$ with

$$\frac{\mathrm{d}a}{\mathrm{d}z}(z,\omega) = -\omega^2 D(z,\omega)\,a(z,\omega). \tag{7.25}$$

Moreover, a derivation as in section 7.1.3 leads to the conclusion that the transmitted time pulse stabilizes around the pulse that follows from the approximation (7.25). We obtain from this the approximation

$$p(z,\tau_\varepsilon + \varepsilon^2 s) \sim [f_0 \star \mathcal{H}](s) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{7.26}$$

where f_0 is defined in (7.19) and τ_ε in (7.24). The pulse-shaping function $\mathcal{H}$ is

$$\mathcal{H}(z,s) = (1/2\pi)\int \exp\left[-\omega^2 \int_0^z \tfrac{1}{4}\gamma_1^2(u)\,\bar{l}(u,\omega)\,\mathrm{d}u\right]\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}\omega.$$

This is the O'Doherty–Anstey pulse-shaping approximation introduced in section 5.2 with $\mathcal{H}$ solving (5.10). Note again that in the *random time frame* the transmitted pressure pulse p is described asymptotically by a *deterministic* pulse shape that is a modification of the pulse in the effective medium case through convolution with a pulse-shaping function, here denoted by $\mathcal{H}$. We discuss the approximation (7.26) in more detail in section 5.2.

7.3. Spherical waves in layered media

In this section we derive the O'Doherty–Anstey approximation for spherical waves in layered media. The approximation for strongly heterogeneous media was first derived in [13]. The rationale for considering the purely layered case is that it sets the stage for solving the locally layered case, which we do in the next section. Although fundamentally different from a global perspective, the scattering process associated with a locally layered medium will resemble locally the one associated with the purely layered case. Thus, we aim at a parametrization of the locally layered case in which the local scattering is captured much as in the layered case discussed below, while global aspects of the wave propagation phenomenon are captured as in the high-frequency approximation discussed in section 4.

We consider the model defined in (3.1)–(3.3) and (3.6) when

$$\gamma_1 \equiv \gamma_1(z). \tag{7.27}$$

The problem differs from those considered in sections 7.1 and 7.2 only in that there we considered an impinging plane-wave pulse, whereas now the source is a spherical wave. The main implication of the assumption (7.27) is that the analysis becomes one dimensional since it allows us to apply the Fourier transform not only with respect to time, but also with respect to the horizontal spatial dimensions. This corresponds to decomposing the source in obliquely travelling plane waves, and for each of these the analysis is much as in the one-dimensional case discussed in sections 7.1 and 7.2. We obtain the leading-order contribution to the Fourier integral over the plane-wave components, the O'Doherty–Anstey pulse-shaping approximation, by a stationary phase argument. The analysis concerns the strongly heterogeneous model. However, the result in the weakly heterogeneous case follows similarly [36].

7.3.1. Decomposition in terms of up- and down-travelling plane waves. We state the version of the ansatz (7.4) and (7.5) that is appropriate for the model at hand. First we transform with respect to time and the horizontal spatial dimensions and express the pressure as

$$p = (1/2\pi\varepsilon) \int \tilde{p}\, \mathrm{e}^{-\mathrm{i}\omega t/\varepsilon}\, \mathrm{d}\omega$$

$$= (1/2\pi\varepsilon) \iiint \hat{p}\, \mathrm{e}^{\mathrm{i}\omega \boldsymbol{\kappa}\cdot\boldsymbol{x}/\varepsilon}\, \mathrm{e}^{-\mathrm{i}\omega t/\varepsilon}\, \mathrm{d}\boldsymbol{\kappa}\, \mathrm{d}\omega. \tag{7.28}$$

Since the medium is purely layered and the amplitudes are independent of the horizontal space argument we obtain the appropriate ansatz as a generalization of (7.4) and (7.5) in the one-dimensional case. We make the ansatz

$$\hat{p} = A\, \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B\, \mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon} \tag{7.29}$$

$$0 = A_z\, \mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B_z\, \mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon} \tag{7.30}$$

where $A = A(z, \boldsymbol{\kappa}, \omega)$, $B = B(z, \boldsymbol{\kappa}, \omega)$ are the amplitudes of the up- and down-travelling wave parts. The phase τ_1 is defined by $\tau_1 = \int_{z_s}^{z} \sqrt{\gamma_1^2(s) - \boldsymbol{\kappa}^2}\, \mathrm{d}s$ with $\boldsymbol{\kappa}^2 \equiv \|\boldsymbol{\kappa}\|_2^2$.

In subsequent sections we will be able to eliminate the up-propagating wave component, B, and obtain an equation for the amplitude A which is a slight modification of (7.12). The transformed pressure, $\hat{p}$, solves for $z \geqslant 0$ the reduced wave equation obtained from (3.1) by elimination of $\boldsymbol{u}$

$$\hat{p}_{zz} + (\omega/\varepsilon)^2([\gamma_1^2(z) - \boldsymbol{\kappa}^2] + \gamma_1^2(z)\, \nu(z/\varepsilon^2))\hat{p} = 0. \tag{7.31}$$

7.3.2. Transport equations. We proceed to obtain the equations for the amplitudes which follow from the ansatz (7.29) and (7.30). Using the same arguments as in section 7.1 we derive equations for the amplitudes that we refer to as transport equations,

$$\begin{aligned} 2\tau_{1,z} A_z + \tau_{1,zz} A &= \mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{A + B\, \mathrm{e}^{-2\mathrm{i}\omega\tau_1/\varepsilon}\} + \tau_{1,zz} B\, \mathrm{e}^{-2\mathrm{i}\omega\tau_1/\varepsilon} \\ 2\tau_{1,z} B_z + \tau_{1,zz} B &= -\mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{A\, \mathrm{e}^{2\mathrm{i}\omega\tau_1/\varepsilon} + B\} + \tau_{1,zz} A\, \mathrm{e}^{2\mathrm{i}\omega\tau_1/\varepsilon}. \end{aligned} \tag{7.32}$$

Here $\tau_{1,z} = \sqrt{\gamma_1(z)^2 - \boldsymbol{\kappa}^2}$ and (7.7) corresponds to the case where $\boldsymbol{\kappa} = \mathbf{0}$. In these transport equations the terms '$\tau_{1,zz} A$' and '$\tau_{1,zz} B$' govern the main behaviour of the solution and give the geometrical effects in the high-frequency approximation, in the deterministic case. Moreover, the stochastic coupling, defined by the terms involving the fluctuations ν, is not purely off-diagonal. This is because the random fluctuations affect the travel time of the propagating pulse. By a change of the dependent variable we obtain transport equations eliminating these effects. Thus, let

$$\alpha = A\, \exp\left[\int_0^z [\tau_{1,zz}/(2\gamma_1) - \mathrm{i}\omega\, \gamma_1 \nu/(2\varepsilon)] \cos(\theta)^{-1}\, \mathrm{d}s\right]$$

$$\beta = B\, \exp\left[\int_0^z [\tau_{1,zz}/(2\gamma_1) + \mathrm{i}\omega\, \gamma_1 \nu/(2\varepsilon)] \cos(\theta)^{-1}\, \mathrm{d}s\right]$$

with the angle θ being defined by

$$\cos(\theta(z, \boldsymbol{\kappa})) = \tau_{1,z}(z, \boldsymbol{\kappa})/\gamma_1(z) = \sqrt{1 - \boldsymbol{\kappa}^2/\gamma_1^2(z)}. \tag{7.33}$$

Then we arrive at the following pair of transport equations:

$$\mathrm{d}\alpha/\mathrm{d}z = \zeta\beta \qquad \mathrm{d}\beta/\mathrm{d}z = \overline{\zeta}\alpha \tag{7.34}$$

with

$$\zeta(z, \boldsymbol{\kappa}, \omega) = \left[\frac{\mathrm{i}\omega\gamma_1(z)\nu(z/\varepsilon^2)}{2\varepsilon\cos(\theta(z, \boldsymbol{\kappa}))} + \frac{\tau_{1,zz}(z, \boldsymbol{\kappa})}{2\gamma_1(z)\cos(\theta(z, \boldsymbol{\kappa}))}\right] \mathrm{e}^{-2\mathrm{i}\omega\tau_\varepsilon(z,\boldsymbol{\kappa})/\varepsilon}$$

and

$$\tau_\varepsilon(z, \boldsymbol{\kappa}) = \int_{z_s}^{z} \gamma_1(s)(1 - \boldsymbol{\kappa}^2/\gamma_1^2(s) + \nu(s/\varepsilon^2)/2)\cos(\theta(s, \boldsymbol{\kappa}))^{-1}\,\mathrm{d}s. \tag{7.35}$$

We eliminate β from the first equation and obtain a stochastic integro-differential equation for the down-going amplitude as before

$$\frac{\mathrm{d}\alpha}{\mathrm{d}z}(z) = -\int_{z}^{\infty} \zeta(z)\,\overline{\zeta(s)}\,\alpha(s)\,\mathrm{d}s \tag{7.36}$$

which is a generalization of (7.12). Here and in the following we suppress the dependence on ω and $\boldsymbol{\kappa}$.

7.3.3. Stabilization and stationary phase evaluation. We next derive an asymptotic expression for the transmitted pressure at an arbitrary point $(\boldsymbol{x}, z)$ in the medium. Recall that the pressure is expressed in terms of the integral (7.28). Based on the stochastic integro-differential equation (7.36) we first obtain an approximation for the wave amplitude A. Upon substitution of this in (7.28) we arrive at an approximation for the transmitted pressure. However, this approximation is in terms of an integral expression over slownesses. We then make use of a stationary phase argument to go from an integral expression over wave components to an expression involving one component only. Furthermore, we show that the resulting approximation is but a slight modification of the high-frequency approximation of the deterministic case. This representation makes explicit the effect of the random modulation of the medium on the transmitted pulse shape and travel time.

Recall that

$$p = (1/2\pi\varepsilon)\iint [A\,\mathrm{e}^{\mathrm{i}\omega\tau_1/\varepsilon} + B\,\mathrm{e}^{-\mathrm{i}\omega\tau_1/\varepsilon}]\,\mathrm{e}^{\mathrm{i}\omega\boldsymbol{\kappa}\cdot\boldsymbol{x}/\varepsilon}\,\mathrm{e}^{-\mathrm{i}\omega t/\varepsilon}\,\mathrm{d}\boldsymbol{\kappa}\,\mathrm{d}\omega. \tag{7.37}$$

As above the reflected signal will be small and

$$p \sim p_{\mathrm{down}} = (1/2\pi\varepsilon)\iiint A\,\mathrm{e}^{\mathrm{i}\omega(S^+ - t)/\varepsilon}\,\mathrm{d}\boldsymbol{\kappa}\,\mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{7.38}$$

where we have introduced the notation

$$S^+ = \boldsymbol{\kappa}\cdot\boldsymbol{x} + \tau_1. \tag{7.39}$$

The phase S^+ solves the eiconal equation (4.11) and is the plane-wave phase in the half-space $z < 0$. Thus, S^+ is the geometric optics phase that corresponds to the effective medium and plane waves incident upon $z > 0$.

We find that for $\overline{\alpha}$ smooth

$$\int_z^\infty E[\zeta(z)\,\overline{\zeta(s)}]\,\overline{\alpha}(s)\,\mathrm{d}s \sim \omega^2\gamma_1^2 l/(4\cos^2(\theta))\,\overline{\alpha}(z) \qquad \text{as} \quad \varepsilon \downarrow 0.$$

Therefore, assuming 'locality' as in the layered case we find from (7.36) that $E[\alpha] \sim a$ as $\varepsilon \downarrow 0$ with a solving for $z > 0$

$$\mathrm{d}a/\mathrm{d}z = -\omega^2\cos(\theta)^{-1} D\, a \tag{7.40}$$

where

$$D = D(z, \boldsymbol{\kappa}) = \gamma_1^2(z) l/(4\cos(\theta(z,\boldsymbol{\kappa}))) \tag{7.41}$$

which is a generalization of (7.16).

Based on the results of the previous sections we expect that an asymptotic expression for p can be obtained by replacing α by a in (7.38). We show that this is indeed the case and also how we can obtain a simple expression for the transmitted pressure pulse by a stationary phase argument. Define a stochastic pressure field by replacing A by a in (7.38)

$$\bar{p} = \iiint \overline{A}\,\exp\left[-\left(\omega^2\int_0^z D\cos(\theta)^{-1}\,\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}\omega(S^+ + \chi_\varepsilon - t)/\varepsilon}\right]\mathrm{d}\boldsymbol{\kappa}\,\mathrm{d}\omega \tag{7.42}$$

with

$$\begin{aligned}\overline{A} &= (2\pi\varepsilon)^{-1}\mathcal{A}\,\exp\left[-\int_0^z [\tau_{1,zz}/(2\gamma_1)]\cos(\theta)^{-1}\,\mathrm{d}s\right]\\ \chi_\varepsilon &= \int_0^z \tfrac{1}{2}\gamma_1\nu\cos(\theta)^{-1}\,\mathrm{d}s\\ \mathcal{A} &= \omega^2\hat{f}(\omega)/(8\pi^2).\end{aligned} \tag{7.43}$$

Note that $D \equiv 0$ and $\chi_\varepsilon \equiv 0$ in the deterministic case when $\nu \equiv 0$. The above formulation corresponds to decomposing the incoming field impinging on the heterogeneous half-space in terms of plane waves. We derive the corresponding expression for $\mathcal{A}$ that is associated with the point source in appendix A of [38].

The variance of $p_{\mathrm{down}} - \bar{p} \sim p - \bar{p}$ is

$$\begin{aligned}E[(p_{\mathrm{down}} - \bar{p})^2](z,t) = \iint \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t/\varepsilon} \iiiint \mathrm{e}^{\mathrm{i}\omega_1 S^+(\boldsymbol{x},z,\boldsymbol{\kappa}_1)/\varepsilon}\,\mathrm{e}^{\mathrm{i}\omega_2 S^+(\boldsymbol{x},z,\boldsymbol{\kappa}_2)/\varepsilon}\\ \times\overline{A}(z,\boldsymbol{\kappa}_1,\omega_1)\,\overline{A}(z,\boldsymbol{\kappa}_2,\omega_2)\,E[\mathcal{R}(z,\boldsymbol{\kappa}_1,\omega_1)\mathcal{R}(z,\boldsymbol{\kappa}_2,\omega_2)]\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\boldsymbol{\kappa}_1\,\mathrm{d}\boldsymbol{\kappa}_2\end{aligned}$$

with

$$\mathcal{R} = \mathrm{e}^{\mathrm{i}\omega\chi_\varepsilon/\varepsilon}\left(\alpha - \exp\left[-\left(\omega^2\int_0^z D\cos(\theta)^{-1}\,\mathrm{d}s\right)\right]\right).$$

It follows from the results in appendix A that

$$E[\mathcal{R}(z,\boldsymbol{\kappa}_1,\omega_1)\,\mathcal{R}(z,\boldsymbol{\kappa}_2,\omega_2)] \sim 0 \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{7.44}$$

and that by dominated convergence

$$E[(p_{\mathrm{down}} - \bar{p})^2] \sim 0 \qquad \text{as} \quad \varepsilon \downarrow 0.$$

Thus the stochastic process $\bar{p}$ is an asymptotic approximation of the transmitted pressure. The expression for $\bar{p}$ in (7.42) can be simplified. Consider the asymptotic evaluation of the integral

$$I = \iint \overline{A}\,\exp\left[-\left(\omega^2\int_0^z D\cos(\theta)^{-1}\,\mathrm{d}s\right)\right]\mathrm{e}^{\mathrm{i}\omega(S^+ + \chi_\varepsilon)/\varepsilon}\,\mathrm{d}\boldsymbol{\kappa}.$$

If we ignore the random perturbation of the phase, that is the χ_ε term, this integral can be evaluated by a standard stationary-phase argument. From the law of the iterated logarithm of probability theory it follows that with probability 1

$$\limsup_{\varepsilon \downarrow 0} |\chi_\varepsilon(\boldsymbol{x}, z, \boldsymbol{\kappa})| \leqslant C\, \varepsilon \sqrt{\log\log \varepsilon^{-1}} \tag{7.45}$$

for some constant $C > 0$. Consequently, to leading order the term χ_ε does not contribute to the phase. In appendix G of [38] we show that we can thus ignore χ_ε when computing the stationary phase point in the method of stationary phase. Therefore,

$$p \sim \int (-\mathrm{i}/\omega)\, \Delta^{-1/2} \mathcal{A} \exp\left[-\int_0^z [\tau_{1,zz}/(2\gamma_1)] \cos(\theta)^{-1}\, \mathrm{d}s\right] \times \exp\left[-\left(\omega^2 \int_0^z D \cos(\theta)^{-1}\, \mathrm{d}s\right)\right] \exp\left[\mathrm{i}\omega(S^+ + \chi_\varepsilon - t)/\varepsilon\right] \mathrm{d}\omega \quad \text{as} \quad \varepsilon \downarrow 0 \tag{7.46}$$

(see [6]). The quantity $\Delta = \Delta(\boldsymbol{x}, z, \boldsymbol{\kappa})$ is the determinant of the Hessian of S^+ with respect to $\boldsymbol{\kappa}$. The above expression is evaluated at the stationary phase point $\overline{\boldsymbol{\kappa}}$ defined as in appendix F of [38]. Modulo the random phase factor χ_ε and the Gaussian spreading factor, expression (7.46) is the high-frequency approximation (4.14), we show this explicitly in appendix E of [38]. Using that

$$[\tau_{1,z} \sqrt{(\mathrm{d}\Omega/\mathrm{d}a)}](\boldsymbol{x}, z) = 1/\sqrt{\Delta(\boldsymbol{x}, z, \overline{\boldsymbol{\kappa}})\, \xi(\boldsymbol{x}, z, \overline{\boldsymbol{\kappa}})}$$
$$\tau_1(\boldsymbol{x}, z) = S^+(\boldsymbol{x}, z, \overline{\boldsymbol{\kappa}})$$

we find

$$p(\boldsymbol{x}, z, \tau_1 + \chi_\varepsilon + \varepsilon s) \sim (4\pi)^{-1}\, \tau_{1,z} \sqrt{(\mathrm{d}\Omega/\mathrm{d}a)(\gamma_0/\gamma_1)}\, [f' \star \mathcal{N}](s) \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{7.47}$$

where $\mathcal{N}$ is the Gaussian distribution of square width V and

$$V = 2 \int_0^z D \cos(\theta)^{-1}\, \mathrm{d}s = (l/2) \int_0^z \left[\gamma_1^2(s)/\cos^2(\theta(s, \overline{\boldsymbol{\kappa}}))\right] \mathrm{d}s.$$

Thus, we have derived the layered version of the result (6.1), which was our objective.

From the approximation (7.47) and (4.14) it follows that the transmitted pulse to leading order can be characterized as the *exact* transmitted pulse associated with the effective medium when this is modified in a similar fashion to the one-dimensional case. If we observe the pulse in a randomly corrected travel time frame then we see to leading order a deterministic pulse that is the transmitted pulse shape in the effective medium modified through convolution with a Gaussian pulse. Note that the width of the Gaussian, V, is large if the ray from the source to the point of observation makes a shallow angle with respect to the vertical, that is, when the propagating pulse experiences random fluctuations with a large correlation length or strong coherence.

Finally, in order to obtain a rigorous argument without explicitly dealing with evanescent modes we need to assume, as in [13], a source that is of compact support in the slowness space $\boldsymbol{\kappa}$ in a neighbourhood of $\bar{\boldsymbol{\kappa}}$.

7.4. Waves in locally layered media

We consider the locally layered model defined in section 3 and derive our main new result the O'Doherty–Anstey pulse-shaping approximation in the strongly heterogeneous case. See [36] for a discussion of the weakly heterogeneous case. The model we consider is thus

$$\begin{aligned} &\rho \boldsymbol{u}_t + \nabla p = \boldsymbol{F}_\varepsilon(\boldsymbol{x}, z, t) \\ &K_\varepsilon^{-1}(\boldsymbol{x}, z)\, p_t + \nabla \cdot \boldsymbol{u} = 0 \end{aligned} \tag{7.48}$$

with

$$\begin{aligned} &\rho(\boldsymbol{x}, z) \equiv \rho_0 \\ &K_\varepsilon^{-1}(\boldsymbol{x}, z) = \begin{cases} K_0^{-1} & z \in (-\infty, 0] \\ K_1^{-1}(\boldsymbol{x}, z)(1 + \nu(z/\varepsilon^2)) & z \in (0, \infty). \end{cases} \end{aligned} \tag{7.49}$$

We start our treatment of (7.48) by deriving generalizations of the transport equations for the harmonic amplitudes given in (7.32). In the purely layered case the amplitudes do not vary horizontally. In the locally layered case we seek a formulation in which they vary only slowly in the horizontal directions. As in the purely layered case, the amplitudes will vary rapidly, on the finest scale of the model, in the depth direction z. This premise that the amplitudes vary slowly horizontally is important in order to obtain simple expressions for the asymptotic approximations of the partial differential equations describing the evolution of the amplitudes.

To motivate our approach in the locally layered case, we now briefly return to the purely layered case.

7.4.1. The purely layered case revisited. Rewrite expression (7.28) for the pressure in the layered case as

$$p = (1/2\pi\varepsilon) \iint [A\, \mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + B\, \mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}]\, \mathrm{e}^{-\mathrm{i}\omega t/\varepsilon}\, \mathrm{d}\boldsymbol{\kappa}\, \mathrm{d}\omega \tag{7.50}$$

where $A = A(z, \boldsymbol{\kappa}, \omega)$, $B = B(z, \boldsymbol{\kappa}, \omega)$. The phase is

$$S^\pm = \boldsymbol{\kappa} \cdot \boldsymbol{x} \pm \tau_1 = \boldsymbol{\kappa} \cdot \boldsymbol{x} \pm \int_{z_s}^{z} \sqrt{\gamma_1^2(s) - \boldsymbol{\kappa}^2}\, \mathrm{d}s$$

and in the homogeneous half-space

$$S^\pm = \boldsymbol{\kappa} \cdot \boldsymbol{x} \pm \sqrt{\gamma_0^2 - \boldsymbol{\kappa}^2}\,(z - z_s). \tag{7.51}$$

The variable $\boldsymbol{\kappa}$ can be interpreted as the horizontal slowness vector of an incoming plane wave. The phases $S^\pm$ solve the eiconal equation associated with the deterministic part of the medium

$$(\nabla S^\pm)^2 = \gamma_1^2 = \rho_0 / K_1(z) \tag{7.52}$$

and are, respectively, up- and down-going plane-wave phases in the homogeneous part of the medium. We refer to $S^\pm$ as generalized plane-wave phases. Thus

$$\hat{p}_{\mathrm{down}} = \iint A\, \mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon}\, \mathrm{d}\boldsymbol{\kappa} \tag{7.53}$$

constitutes a decomposition of the impinging pulse in terms of obliquely travelling generalized plane waves. A possible ray configuration for S^+ is shown in figure 7, rays are denoted by Γ^+. The phase fronts will be orthogonal to these. We also have the relation $S^-(\boldsymbol{x}, z, \boldsymbol{\kappa}) \equiv -S^+(\boldsymbol{x}, z, -\boldsymbol{\kappa})$. Thus, if we change the orientation of the rays in figure 7, we obtain the

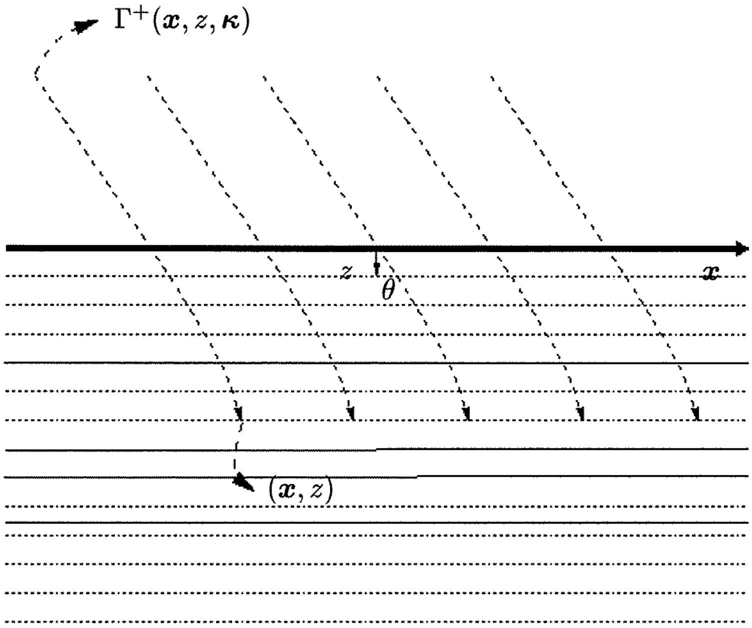

Figure 7. The geometrical optics rays associated with a particular incoming plane wave, a particular κ. In the homogeneous half-space they are linear and parallel, in the heterogeneous layered half-space they remain parallel but are curved. The rays are denoted by $\Gamma^+(x, z, \kappa)$ and the angle between these and the vertical is θ.

rays associated with $S^-(x, z, -\kappa)$. The formulation (7.50) can be seen as a generalization of the high-frequency ansatz (4.10) in that we have included the reflected field. In contrast to the usual high-frequency asymptotics, an analysis based on this representation will capture the modulation of the propagating pulse due to a local random coupling between the up- and down-propagating wave fields, the scattering process which induces the pulse modulation we want to characterize. In the purely layered case, the problem decouples and becomes essentially one dimensional. The physical interpretation of this is that because of ray symmetry only amplitude pairs with the same horizontal slowness κ at the surface interact through scattering (see figure 8). Equivalently, we can consider each Fourier component $\hat{p}$ in isolation. The ray denoted by Γ^- in figure 8 is associated with S^- and couples with the ray Γ^+ for a horizontal interface according to Snell's law of reflection. Next, we show how this picture generalizes in the locally layered case.

7.4.2. Decomposition in terms of generalized plane waves. We seek to generalize the transport equations (7.32) to the locally layered case. We retain the parametrization (7.50) for the time-transformed pressure

$$\hat{p} = \iint [A\, e^{i\omega S^+/\varepsilon} + B\, e^{i\omega S^-/\varepsilon}]\, d\kappa \tag{7.54}$$

but hasten to point out that this is not a standard representation since the medium now is varying horizontally. This new representation of the pressure pulse enables us to generalize the analysis concerning a layered medium to the locally layered case. The phases S^+ and S^- in (7.54) are defined as for a layered medium. They solve the eiconal equation (7.52) with $K_1 = K_1(x, z)$ and with initial conditions at the surface defined by (7.51). However, because the medium

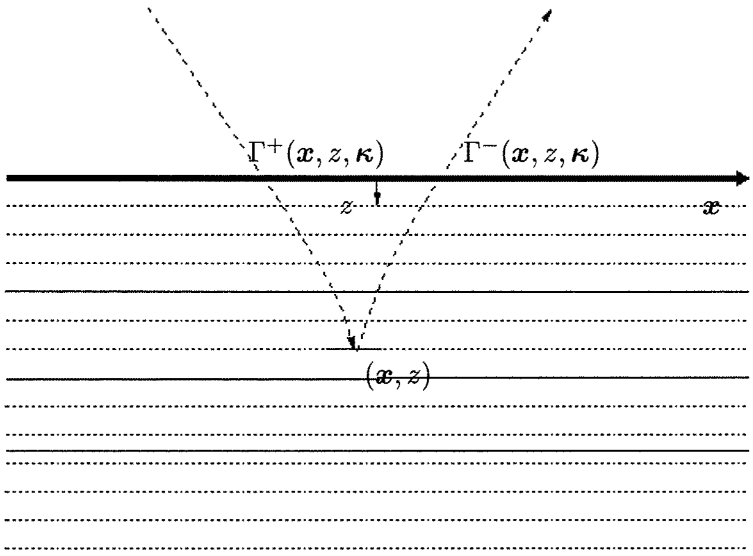

Figure 8. An incoming geometrical optics ray is reflected according to Snell's law. The medium is purely layered and the angles with respect to the vertical are equal for the two rays shown. Note that the two rays are associated with the same $\boldsymbol{\kappa}$, and this defines the horizontal slowness at the surface.

parameters vary horizontally, the associated rays will not be parallel in the half-space $z > 0$ as they are in the purely layered case illustrated in figure 7. Furthermore, the general ray picture indicates that the amplitudes A and B also depend on the horizontal argument and that wave components with different horizontal slowness vectors at the surface *interact* as they propagate. The amplitudes will vary in general on the scale of the local scattering, the scale ε^{-2}, though according to our formulation only in the z-direction.

As before we need to complement the ansatz (7.54) with an additional constraint. We make the ansatz

$$\hat{p} = \iint [A\, \mathrm{e}^{\mathrm{i}\omega S^{+}/\varepsilon} + B\, \mathrm{e}^{\mathrm{i}\omega S^{-}/\varepsilon}]\, \mathrm{d}\boldsymbol{\kappa} \tag{7.55}$$

$$0 = \iint [A_z\, \mathrm{e}^{\mathrm{i}\omega S^{+}/\varepsilon} + B_z\, \mathrm{e}^{\mathrm{i}\omega S^{-}/\varepsilon}]\, \mathrm{d}\boldsymbol{\kappa} \tag{7.56}$$

with $A = A(\boldsymbol{x}, z, \boldsymbol{\kappa}, \omega)$ and $B = B(\boldsymbol{x}, z, \boldsymbol{\kappa}, \omega)$. The Fourier-transformed pressure, $\hat{p}$, solves for $z \geqslant 0$ the reduced wave equation obtained from (3.1) by elimination of $\boldsymbol{u}$

$$\Delta\hat{p} + (\omega/\varepsilon)^2 \gamma_1^2 (1 + \nu)\hat{p} = 0. \tag{7.57}$$

Note the integration over slownesses in (7.55) and (7.56). The lateral phase variation does not correspond to the Fourier basis as in the layered case, thus the wave components now interact. The situation differs from that of the previous section where decoupling of plane-wave components made it possible to obtain asymptotic approximations for the amplitudes by standard techniques, using stochastic ordinary differential equations. To cope with the more general scattering picture we introduce a mapping in the slowness domain. This enables us to derive stochastic integro-differential equations for the amplitudes, similar to those of the previous section.

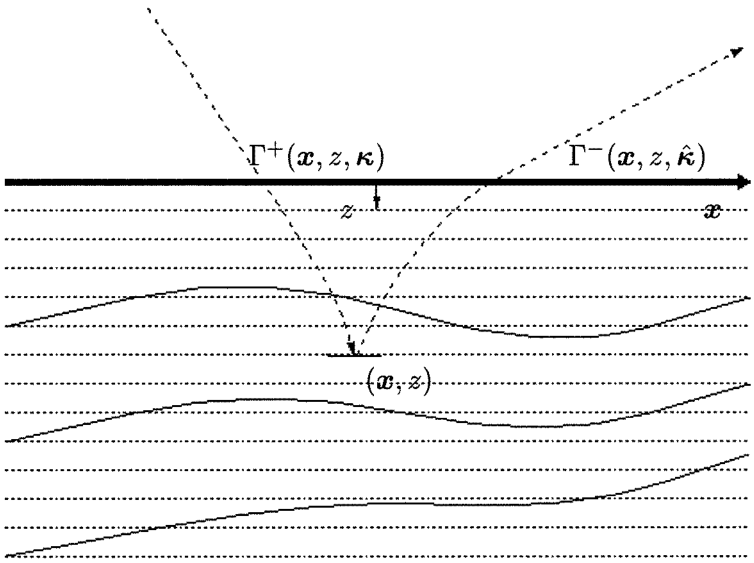

Figure 9. Ray coupling through slowness mapping for a locally layered medium. The ray geometry is a generalization of the geometry shown in figure 8. The rays still satisfy Snell's law of reflection with respect to a horizontal interface. However, due to the lateral variation in the medium parameters, the rays involved have a different horizontal slowness at the surface, corresponding to different $\boldsymbol{\kappa}$. The slowness mapping articulates this coupling of rays associated with different $\boldsymbol{\kappa}$.

7.4.3. Mapping of the slowness vector. In the locally layered case the rays associated with a given surface slowness vector will in general form a complicated ray pattern. The rays, defined as the characteristic directions associated with the solution of the eiconal equation, are parallel at the surface but do not remain so inside the medium. Recall that the fine-scale modulation of the compliance is a function of the depth variable z only. Hence, at a fixed depth, scattering couples the up- and down-going modes whose ray paths have angles of incidence with respect to the z-direction which are equal and coplanar (see figure 9 for a two-dimensional example). If the down-going ray path corresponds to the surface slowness vector $\boldsymbol{\kappa}$, we denote the slowness vector corresponding to the reflected path satisfying this law of reflection by $\hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)$. The mapping is a function of the space argument. A local existence proof of the mapping is given in appendix B. In the following we will make use of the following notation for a function $f = f(\boldsymbol{x}, z, \boldsymbol{\kappa})$ evaluated at the image/inverse image of $\boldsymbol{\kappa}$:

$$\hat{f} = f(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z))$$

$$\check{f} = f(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}^{-1}(\boldsymbol{\kappa}; \boldsymbol{x}, z)).$$

From the definition of this mapping it follows that $\nabla_\perp S^+ = \widehat{\nabla_\perp S^-}$ and $S_z^+ = -\widehat{S_z^-}$, and that $\hat{\boldsymbol{\kappa}}^{-1}(\boldsymbol{\kappa}; \boldsymbol{x}, z) = -\hat{\boldsymbol{\kappa}}(-\boldsymbol{\kappa}; \boldsymbol{x}, z)$. Here, $\nabla_\perp$ denotes the horizontal gradient. Figure 9 illustrates how we have generalized the ray picture of figure 8 in the purely layered case by introducing a mapping in the slowness domain. This scattering picture is an idealization of the one we use in [36], but captures the essential aspects of the ray geometry needed to describe the evolution of the front.

7.4.4. Coupling in transport equations. In this section we obtain the equations for the amplitudes which follow from the ansatz (7.55) and (7.56). In doing this we will make use of the mapping for the slownesses defined above. First we substitute (7.55) in (7.57) to obtain

$$\iint [\{2\nabla S^+ \cdot \nabla A + \Delta S^+ A - \mathrm{i}(\varepsilon/\omega)\Delta A\}\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + \{2\nabla S^- \cdot \nabla B + \Delta S^- B - \mathrm{i}(\varepsilon/\omega)\Delta B\}\mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}$$
$$-\mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{A\,\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + B\,\mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}\}]\,\mathrm{d}\boldsymbol{\kappa} = 0.$$

From (7.56) we find

$$\iint [\{S_z^+ A_z\,\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + S_z^- B_z\,\mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}\} - \mathrm{i}\varepsilon/\omega\{A_{zz}\,\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + B_{zz}\,\mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}\}]\,\mathrm{d}\boldsymbol{\kappa} = 0.$$

We next combine the above two integral relations

$$\iint [\{S_z^+ A_z + 2\nabla_\perp S^+ \cdot \nabla_\perp A + \Delta S^+ A - \mathrm{i}(\varepsilon/\omega)\Delta_\perp A\}\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon}$$
$$+\{S_z^- B_z + 2\nabla_\perp S^- \cdot \nabla_\perp B + \Delta S^- B - \mathrm{i}(\varepsilon/\omega)\Delta_\perp B\}\mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}$$
$$-\mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{A\,\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + B\,\mathrm{e}^{\mathrm{i}\omega S^-/\varepsilon}\}]\,\mathrm{d}\boldsymbol{\kappa} = 0. \tag{7.58}$$

In the standard high-frequency approximation we can solve for each wave component separately. For the model at hand the 'high-frequency' fluctuations in the medium parameters cause a coupling between components. To articulate this coupling we rewrite (7.56) and (7.58) as

$$\iint [\{S_z^+ A_z + 2\nabla_\perp S^+ \cdot \nabla_\perp A + \Delta S^+ A - \mathrm{i}(\varepsilon/\omega)\Delta_\perp A\}\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon}$$
$$+\{\widehat{S^-}_z \hat{B}_z + 2\widehat{\nabla_\perp S^-} \cdot \widehat{\nabla_\perp B} + \widehat{\Delta S^-}\hat{B} - \mathrm{i}(\varepsilon/\omega)\widehat{\Delta_\perp B}\}\mathrm{e}^{\mathrm{i}\omega \hat{S}^-/\varepsilon}\, J$$
$$-\mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{A\,\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + \hat{B}\,\mathrm{e}^{\mathrm{i}\omega \hat{S}^-/\varepsilon}\, J\}]\,\mathrm{d}\boldsymbol{\kappa} = 0$$

$$\iint [A_z\,\mathrm{e}^{\mathrm{i}\omega S^+/\varepsilon} + \hat{B}_z\,\mathrm{e}^{\mathrm{i}\omega \hat{S}^-/\varepsilon}\, J]\,\mathrm{d}\boldsymbol{\kappa} = 0$$

with J denoting the Jacobian of the transformation $\boldsymbol{\kappa} \mapsto \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)$

$$J(\boldsymbol{x}, z, \boldsymbol{\kappa}) = |\partial\hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)/\partial\boldsymbol{\kappa}|.$$

At this point we require the integral kernels of the above two integral relations to be zero, since then the appropriate local interaction between up- and down-propagating components are enforced. By adding/subtracting a multiple of $S_z^+ (= -\hat{S}_z^-)$ times the second kernel to/from the first we obtain the generalized transport equations,

$$2\nabla S^+ \cdot \nabla A + \Delta S^+ A - \mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{A + \hat{B}\,\mathrm{e}^{\mathrm{i}\omega(\hat{S}^- - S^+)/\varepsilon}\, J\}$$
$$= \mathrm{i}(\varepsilon/\omega)\Delta_\perp A - \hat{R}^-\,\mathrm{e}^{\mathrm{i}\omega(\hat{S}^- - S^+)/\varepsilon}\, J \tag{7.59}$$

$$2\nabla S^- \cdot \nabla B + \Delta S^- B - \mathrm{i}(\omega/\varepsilon)\gamma_1^2\, \nu\{B + \check{A}\,\mathrm{e}^{\mathrm{i}\omega(\check{S}^+ - S^-)/\varepsilon}\, \check{J}^{-1}\}$$
$$= \mathrm{i}(\varepsilon/\omega)\Delta_\perp B - \check{R}^+\,\mathrm{e}^{\mathrm{i}\omega(\check{S}^+ - S^-)/\varepsilon}\, \check{J}^{-1} \tag{7.60}$$

with

$$R^+ = 2\nabla_\perp S^+ \cdot \nabla_\perp A + \Delta S^+ A - \mathrm{i}(\varepsilon/\omega)\Delta_\perp A \tag{7.61}$$

$$R^- = 2\nabla_\perp S^- \cdot \nabla_\perp B + \Delta S^- B - \mathrm{i}(\varepsilon/\omega)\Delta_\perp B. \tag{7.62}$$

This is a generalization of (7.32). If we compare the above transport equations with the corresponding ones in the high-frequency case given in (4.12), we see that, apart from a stochastic coupling, the transport equation for A has been changed only in that the horizontal rather than the full Laplacian of A appears. By considering the amplitude pair rather than a forward-propagating component only, we have eliminated the component of the Laplacian in the direction in which the microscale structure will cause the amplitudes to be rapidly varying.

In these transport equations the terms '$\Delta S^+ A$' and '$\Delta S^- B$' govern the main behaviour of the solution and contain the geometrical spreading effect in the high-frequency effective medium approximation. As in the layered case the stochastic coupling, defined by the terms involving the fluctuations ν, is not purely off-diagonal, since the random fluctuations affect the travel time of the propagating pulse. By a change of the dependent variable we next obtain transport equations where these effects have been compensated for,

$$\alpha = A\,\exp\left[\int_{\Gamma^+} [\Delta S^+/(2\gamma_1) - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\,\mathrm{d}s\right]$$

$$\beta = B\,\exp\left[\int_{\Gamma^-} [\Delta S^-/(2\gamma_1) - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\,\mathrm{d}u\right].$$

The paths $\Gamma^\pm$ are the geometrical optics ray paths associated with $S^\pm$ (see figure 9).

We thus arrive at the following pair of transport equations:

$$\begin{aligned}
\mathrm{d}\alpha/\mathrm{d}s &= \mathrm{i}(\omega/2\varepsilon)\gamma_1\nu\,\hat{\beta}\,\mathrm{e}^{\Phi^+ - \hat{\Phi}^-}\,J \\
&\quad + \{\mathrm{i}(\varepsilon/\omega)\Delta_\perp A\,\mathrm{e}^{\mathrm{i}\omega S^+\varepsilon^{-1}+\Phi^+} - \hat{R}^-\,\mathrm{e}^{\mathrm{i}\omega\hat{S}^-\varepsilon^{-1}+\Phi^+}\,J\}/(2\gamma_1) \\
\mathrm{d}\beta/\mathrm{d}u &= \mathrm{i}(\omega/2\varepsilon)\gamma_1\nu\,\check{\alpha}\,\mathrm{e}^{-\check{\Phi}^+ + \Phi^-}\,\check{J}^{-1} \\
&\quad + \{\mathrm{i}(\varepsilon/\omega)\Delta_\perp B\,\mathrm{e}^{\mathrm{i}\omega S^-\varepsilon^{-1}+\Phi^-} - \check{R}^+\,\mathrm{e}^{\mathrm{i}\omega\check{S}^+\varepsilon^{-1}+\Phi^-}\,\check{J}^{-1}\}/(2\gamma_1)
\end{aligned} \tag{7.63}$$

with

$$\begin{aligned}
\Phi^+ &= -\mathrm{i}\omega S^+/\varepsilon + \int_{\Gamma^+} [\Delta S^+/(2\gamma_1) - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\,\mathrm{d}s \\
\Phi^- &= -\mathrm{i}\omega S^-/\varepsilon + \int_{\Gamma^-} [\Delta S^-/(2\gamma_1) - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\,\mathrm{d}u
\end{aligned} \tag{7.64}$$

and with s and u the arc lengths along the characteristic ray paths Γ^+ and Γ^-, respectively.

7.4.5. Leading-order transport equations. In the purely layered case the bracketed terms on the right-hand sides of (7.63) are asymptotically negligible. In [36] we argue that they do not contribute to the leading-order asymptotic approximation of the transmitted pressure in the locally layered case. Thus, retaining the notation for the amplitudes we arrive at the following approximate transport equations:

$$\begin{aligned}
\mathrm{d}\alpha/\mathrm{d}s &= \mathrm{i}(\omega/2\varepsilon)\gamma_1\nu\,\hat{\beta}\,\mathrm{e}^{(\Phi^+ - \hat{\Phi}^-)}\,J \\
\mathrm{d}\beta/\mathrm{d}u &= \mathrm{i}(\omega/2\varepsilon)\gamma_1\nu\,\check{\alpha}\,\mathrm{e}^{(-\check{\Phi}^+ + \Phi^-)}\,\check{J}^{-1}.
\end{aligned} \tag{7.65}$$

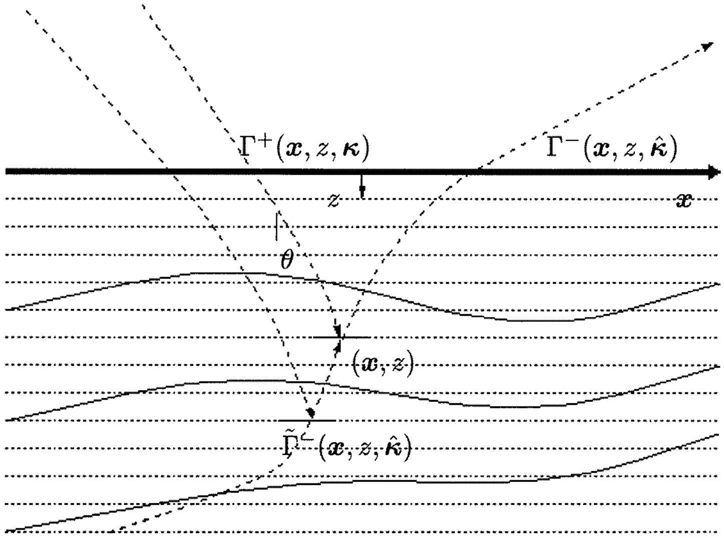

Figure 10. The rays involved in the integro-differential equation (7.66). An incoming ray Γ^+ in the locally layered medium couples to a reflected ray Γ^- through the slowness mapping.

In a manner similar to the one in section 7.3 we can eliminate β from the first equation and obtain a generalization of (7.36)

$$\mathrm{d}\alpha(\boldsymbol{x}, z, \boldsymbol{\kappa}, \omega)/\mathrm{d}s = -(\omega/2\varepsilon)^2 \int_{\tilde{\Gamma}^-} [\gamma_1(\boldsymbol{x}, z)\, \gamma_1(\boldsymbol{x}(u), z(u))\, \nu(z/\varepsilon^2)\, \nu(z(u)/\varepsilon^2) \\ \times J(\boldsymbol{x}, z, \boldsymbol{\kappa})\, J^{-1}(\boldsymbol{x}(u), z(u), \tilde{\boldsymbol{\kappa}}(u))\, \mathrm{e}^{\tilde{\Phi}(u)}]\, \alpha(\boldsymbol{x}(u), z(u), \tilde{\boldsymbol{\kappa}}(u), \omega)\, \mathrm{d}u. \qquad (7.66)$$

Here $\tilde{\Gamma}^- = \tilde{\Gamma}^-(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z))$ is the semi-infinite characteristic ray segment, one of the rays associated with S^- and the slowness $\hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)$, that terminates at $(\boldsymbol{x}, z)$, see figure 10, and u is arc length along this path. The layered case of the previous section corresponds to $J \equiv 1$. We use the notation

$$\tilde{\boldsymbol{\kappa}}(u) \equiv \hat{\boldsymbol{\kappa}}^{-1}(\hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z); \boldsymbol{x}(u), z(u))$$

$$\tilde{\Phi}(u) \equiv \Phi^+(\boldsymbol{x}, z, \boldsymbol{\kappa}) - \Phi^+(\boldsymbol{x}(u), z(u), \tilde{\boldsymbol{\kappa}}(u)) \\ -\Phi^-(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)) + \Phi^-(\boldsymbol{x}(u), z(u), \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)).$$

It follows from the definitions that $\tilde{\boldsymbol{\kappa}}(0) = \boldsymbol{\kappa}$ and $\tilde{\Phi}(0) = 0$ with $(\boldsymbol{x}(0), z(0)) = (\boldsymbol{x}, z)$.

7.4.6. Stabilization of the pulse. In order to facilitate comparison with the layered case (7.36) we write (7.66) as

$$\frac{\mathrm{d}\alpha(\boldsymbol{y})}{\mathrm{d}s} = \int_\Gamma K_\varepsilon(\boldsymbol{y}, \boldsymbol{y}(u))\, \alpha(\boldsymbol{y}(u))\, \mathrm{d}u$$

with

$$\boldsymbol{y} \equiv (\boldsymbol{x}, z, \boldsymbol{\kappa})$$

$$\boldsymbol{y}(u) \equiv (\boldsymbol{x}(u), z(u), \tilde{\boldsymbol{\kappa}}(u))$$

$$K_\varepsilon(\boldsymbol{y}, \boldsymbol{y}(u)) = -(\omega/2\varepsilon)^2[\gamma_1(\boldsymbol{x}, z)\, \gamma_1(\boldsymbol{x}(u), z(u))\, \nu(z/\varepsilon^2)\, \nu(z(u)/\varepsilon^2) \\ \times J(\boldsymbol{x}, z, \boldsymbol{\kappa})\, J^{-1}(\boldsymbol{x}(u), z(u), \tilde{\boldsymbol{\kappa}}(u))\, \mathrm{e}^{\tilde{\Phi}(u)}]$$

and $\tilde{\boldsymbol{\kappa}}(u)$, J and $\tilde{\Phi}(u)$ defined as in (7.66) and where we suppress the ω dependence. The path Γ is $\Gamma(u) = \boldsymbol{y}(u)$ such that its projection on the space coordinates is $\tilde{\Gamma}^-$ and its projection on the subspace $\boldsymbol{\kappa}$ is $\tilde{\boldsymbol{\kappa}}(u)$, where u is the arc length along $\tilde{\Gamma}^-$.

As in the layered case we find that for $\overline{\alpha}$ smooth

$$\int_\Gamma E[K_\varepsilon(\boldsymbol{y}, \boldsymbol{y}(u))]\,\overline{\alpha}(\boldsymbol{y}(u))\,\mathrm{d}u \sim -\omega^2 D(\boldsymbol{y})\,\overline{\alpha}(\boldsymbol{y}) \qquad \text{as} \quad \varepsilon \downarrow 0$$

where

$$D = \gamma_1^2\, l/(4\cos(\theta)) \tag{7.67}$$

and with θ being the angle between Γ^+ and the vertical direction at $(\boldsymbol{x}, z)$, shown in figure 10. We have assumed that the path $\Gamma^+(\boldsymbol{x}, z, \boldsymbol{\kappa})$ is nowhere horizontal, hence $\cos(\theta) > 0$. If we assume 'locality' as in the layered case, we find that $E[\alpha] \sim a$ as $\varepsilon \downarrow 0$, with a solving for $z > 0$

$$\mathrm{d}a/\mathrm{d}s = -\omega^2 D\, a. \tag{7.68}$$

As above, the transmitted pressure can be represented approximately by

$$p \sim (1/2\pi\varepsilon) \iiint A\, \mathrm{e}^{\mathrm{i}\omega(S^+ - t)/\varepsilon}\, \mathrm{d}\boldsymbol{\kappa}\, \mathrm{d}\omega \qquad \text{as} \quad \varepsilon \downarrow 0.$$

In view of previous analysis we define an approximation for p by replacing α by a as

$$\bar{p} = \iiint \overline{A}\, \exp\left[-\left(\omega^2 \int_{\Gamma^*} D\, \mathrm{d}s\right)\right] \mathrm{e}^{\mathrm{i}\omega(S^+ + \chi_\varepsilon - t)/\varepsilon}\, \mathrm{d}\boldsymbol{\kappa}\, \mathrm{d}\omega \tag{7.69}$$

with $\overline{A}$ generalizing (7.43)

$$\overline{A} = (2\pi\varepsilon)^{-1} \mathcal{A}\, \exp\left[-\int_{\Gamma^*} \Delta S^+/(2\gamma_1)\, \mathrm{d}s\right]$$

$$\chi_\varepsilon = \int_{\Gamma^*} \tfrac{1}{2}\gamma_1 \nu\, \mathrm{d}s$$

$$\mathcal{A} = \omega^2 \hat{f}(\omega)/(8\pi^2).$$

Here Γ^* is the part of the ray segment between the source and the point of observation that is located in the half-space $z \geqslant 0$. Note that (7.69) is an exact analogue of (7.42), and differs only in that the phase S^+ and the associated rays Γ^* are defined more generally. By an argument analogous to the one in the layered case we therefore also obtain an approximation for the transmitted pulse in the locally layered case.

Consider problem (3.1). Then for $z > 0$ with probability one the transmitted pressure admits the asymptotic characterization

$$p(\boldsymbol{x}, z, \tau_1 + \chi_\varepsilon + \varepsilon s) \sim (4\pi)^{-1}\, \tau_{1,z} \sqrt{(\mathrm{d}\Omega/\mathrm{d}a)(\gamma_0/\gamma_1)}\, [f' \star \mathcal{N}](s) \qquad \text{as} \quad \varepsilon \downarrow 0$$

where $\mathrm{d}\Omega$ is an element of solid angle of the initial direction of rays going from the source to the point of observation and $\mathrm{d}a$ is the associated element of area on the wavefront (figure 2 illustrates this). The function $\mathcal{N}$ is the centred Gaussian distribution of square width V, where

$$V = (l/2) \int_{\Gamma^*} \gamma_1^2 \cos(\theta)^{-1}\, \mathrm{d}s$$

$$\chi_\varepsilon = \int_{\Gamma^*} \tfrac{1}{2}\gamma_1 \nu\, \mathrm{d}s$$

$$l \equiv \int_0^\infty E[v(0)v(s)]\,\mathrm{d}s$$

$$\gamma_1 = \sqrt{\rho_0/K_1}.$$

This is the result stated in section 6. It requires a generalization to functionals of the limit argument presented in appendix A. This is carried out in [36]. The argument is based on the above-mentioned assumptions on the ray geometry, that there are no confluence of rays, no caustics. Moreover, as in the layered case, to avoid dealing with evanescent modes the Fourier content of the source is confined to a neighbourhood of the stationary phase slowness $\overline{\boldsymbol{\kappa}}$.

8. Conclusions

We have generalized the O'Doherty–Anstey pulse-shaping approximation to a locally layered medium. This is a medium with smooth deterministic three-dimensional background variations modulated by random laminated fluctuations, which need not be plane on the macroscale nor differentiable on the microscale. The O'Doherty–Anstey approximation determines how the random fluctuations affect a propagating acoustic pulse generated by a point source. The pulse is affected in two ways. First, its arrival time is itself random. Second, when the pulse is observed relative to its random arrival time, it is to leading-order deterministic, and is the convolution of the transmitted pulse through the deterministic background medium with a pulse-shaping function. The parameters that determine the pulse-shaping function are given by an integral of medium statistics over the geometrical optics ray path that emanates from the source point and goes through the point of observation.

The O'Doherty–Anstey approximation is obtained by using a generalization of geometrical optics. This generalization incorporates a propagating as well as a reflected wave field and enables us to extend the analysis for a purely layered medium to a locally layered medium. We also review here the O'Doherty–Anstey approximation in the layered case and introduce a new interpretation for it in terms of the distribution of a random sum.

We consider only transmitted pulses. Physically, it is clear that a corresponding pulse-shaping approximation can be constructed for a pulse reflected from a discontinuity in the background medium. The path integral of medium statistics that determines the pulse-shaping function becomes in this case the geometrical optics ray path that emanates from the source, is *reflected* from the interface and then goes through the point of observation. A rigorous derivation of this result is, however, rather complicated.

In addition to the analysis of reflected pulses, future work will consider *sets* of transmitted or reflected signals and how these are correlated. Such information can be used in the solution of inverse problems in seismic imaging and remote sensing. In [36, 37] we illustrate how this can be done in a very simple context involving a layered medium.

Appendix A. Limit result, layered case

We verify (7.44). This result follows from

$$E[\alpha_1(z)] \sim \mathrm{e}^{-\omega_1^2 V_1(z)}\alpha_1(0) \tag{A.1}$$

$$E[\alpha_1(z)\,\mathrm{e}^{\mathrm{i}\Phi(z)/\varepsilon}] \sim E[\alpha_1(z)]E[\mathrm{e}^{\mathrm{i}\Phi(z)/\varepsilon}] \tag{A.2}$$

$$E[\alpha_1(z)\,\alpha_2(z)\,\mathrm{e}^{\mathrm{i}\Phi(z)/\varepsilon}] \sim E[\alpha_1(z)]E[\alpha_2(z)]E[\mathrm{e}^{\mathrm{i}\Phi(z)/\varepsilon}] \qquad \text{as} \quad \varepsilon \downarrow 0. \tag{A.3}$$

Here $\alpha_i(z) \equiv \alpha(z, \boldsymbol{\kappa}_i, \omega_i)$, $V_i(z) \equiv \int_0^z D(s, \boldsymbol{\kappa}_i)\cos(\theta(s, \boldsymbol{\kappa}_i))^{-1}\,\mathrm{d}s$ where θ and D are defined, respectively, by (7.33) and (7.41). Also $\Phi(z) \equiv \omega_1\chi_\varepsilon(z, \boldsymbol{\kappa}_1) + \omega_2\chi_\varepsilon(z, \boldsymbol{\kappa}_2)$, where χ_ε is defined in (7.43). We show only (A.1) and (A.3), equation (A.2) follows by a similar argument.

We will make use of the following result [10]. Let X^ε be a finite-dimensional state vector satisfying

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}z} = \varepsilon^{-1}F(z, \nu(z/\varepsilon^2), \tau(z)/\varepsilon, X^\varepsilon) + G(z, \nu(z/\varepsilon^2), \tau(z)/\varepsilon, X^\varepsilon) \tag{A.4}$$

$$X^\varepsilon|_{z=L} = X_L \tag{A.5}$$

with ν a random, mean zero, stationary process with rapidly decaying correlation function and X_L a *deterministic* 'end' condition. The dependence on τ is through a periodic function and $\partial_z\tau \neq 0$. Also $E[F(z, \nu(s), \tau, X)] = 0$ for all z, s, τ, X.

Define the operator

$$B^j(z, X) = \lim_{Y\to\infty} Y^{-1}\int_0^Y\int_0^\infty -E[F(z, \nu(0), y, X)\cdot\nabla_X F^j(z, \nu(r), y, X)]\,\mathrm{d}r\,\mathrm{d}y$$
$$+ \lim_{Y\to\infty} Y^{-1}\int_0^Y E[G^j(z, \nu(0), y, X)]\,\mathrm{d}y. \tag{A.6}$$

If it is linear, $B^j(z, X) = b^j(z)\cdot X$, then

$$\frac{\mathrm{d}\langle X\rangle}{\mathrm{d}z} = B(z)\langle X\rangle \tag{A.7}$$

with the rows of B being b^j and $\langle X\rangle = E[X]$.

In order to apply the above result we use an invariant embedding approach. We assume that the medium parameters are constant for $z > L$. Because of the finite speed of propagation we can do this without affecting the solution over a finite time frame.

Recall equations (7.32) for the amplitudes. Here we make the change of variables

$$\alpha = A\,\exp\left[\int_0^z [\tau_{1,zz}/(2\gamma_1) - \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\cos(\theta)^{-1}\,\mathrm{d}s\right]$$

$$\beta = B\,\exp\left[\int_0^z [\tau_{1,zz}/(2\gamma_1) + \mathrm{i}(\omega/\varepsilon)\gamma_1\nu/2]\cos(\theta)^{-1}\,\mathrm{d}s\right]\exp[-2\mathrm{i}\omega\tau(0)/\varepsilon - \mathrm{i}\omega T(0)]$$

with

$$\tau(z) = \int_z^L \sqrt{\gamma_1(s)^2 - \boldsymbol{\kappa}^2}\,\mathrm{d}s$$

$$T(z) = \varepsilon^{-1}\int_z^L \gamma_1(s)\,\nu(s/\varepsilon^2)\cos(\theta(s))^{-1}\,\mathrm{d}s.$$

Observe that β differs from that defined in (7.34) by a phase factor. The resulting amplitude equations are

$$\mathrm{d}\alpha/\mathrm{d}z = \zeta\beta \qquad \mathrm{d}\beta/\mathrm{d}z = \overline{\zeta}\alpha \tag{A.8}$$

with

$$\zeta(z) = \cos(\theta)^{-1}[\mathrm{i}\omega\gamma_1\nu(z/\varepsilon^2)/(2\varepsilon) + \tau_{1,zz}/(2\gamma_1)]\,\mathrm{e}^{\mathrm{i}\omega(2\tau/\varepsilon+T)}$$
$$\equiv (\mathrm{i}\nu f/\varepsilon + g)e(\omega)$$

where we defined

$$e(\omega) = \mathrm{e}^{\mathrm{i}\omega(2\tau/\varepsilon + T)}.$$

We first show (A.1). Define

$$\Gamma(z, \boldsymbol{\kappa}, \omega) = \beta(z, \boldsymbol{\kappa}, \omega)/\alpha(z, \boldsymbol{\kappa}, \omega)$$

$$\Theta(z, \boldsymbol{\kappa}, \omega) = \alpha(L, \boldsymbol{\kappa}, \omega)/\alpha(z, \boldsymbol{\kappa}, \omega)$$

and let the state vector be $X = [\Gamma, \Theta, T]$. Then

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}z} &= \overline{\zeta} - \zeta\Gamma^2 \\
&= -(\mathrm{i}\nu f/\varepsilon)\,[e(-\omega) + e(\omega)\Gamma^2] + g\,[e(-\omega) - e(\omega)\Gamma^2] \\
\frac{\mathrm{d}\Theta}{\mathrm{d}z} &= -\zeta\Theta\Gamma \\
&= -(\mathrm{i}\nu f/\varepsilon)\,e(\omega)\Theta\Gamma - g\,e(\omega)\Theta\Gamma \\
\frac{\mathrm{d}T}{\mathrm{d}z} &= -2f\,\nu/(\varepsilon\omega).
\end{aligned}$$

with the end conditions $\Gamma(L) = 0$, $\Theta(L) = 1$ and $T(L) = 0$. In view of (A.6) we find that the drift operator B associated with the above system is linear, moreover it is *diagonal*. In particular,

$$\frac{\mathrm{d}\langle\Theta\rangle}{\mathrm{d}z} = f^2 l\langle\Theta\rangle \tag{A.9}$$

from which (A.1) follows.

Next we show (A.3). Define

$$J(z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \omega_1, \omega_2) = \Theta(z, \boldsymbol{\kappa}_1, \omega_1)\,\Theta(z, \boldsymbol{\kappa}_2, \omega_2)\,\Psi(z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \omega_1, \omega_2)$$

with

$$\Psi(z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \omega_1, \omega_2) = \exp[\mathrm{i}(\Phi(L) - \Phi(z))/\varepsilon] = \exp\left[\mathrm{i}\int_z^L [\nu(f_1 + f_2)/\varepsilon]\,\mathrm{d}s\right]$$

and the subscript j indicating that the function is evaluated at $(\boldsymbol{\kappa}_j, \omega_j)$. Let the state vector be $X = [J, \Psi, \Theta_1, \Theta_2, \Gamma_1, \Gamma_2, T_1, T_2]$. Then

$$\begin{aligned}
\frac{\mathrm{d}J}{\mathrm{d}z} &= -\mathrm{i}\nu/\varepsilon\sum_{j=1}^{2}[f_j e(\omega_j)\Gamma_j + f_j]J - \sum_{j=1}^{2}[g_j e(\omega_j)\Gamma_j]J \\
\frac{\mathrm{d}\Psi}{\mathrm{d}z} &= -\mathrm{i}\nu(f_1 + f_2)/\varepsilon\,\Psi \\
\frac{\mathrm{d}\Gamma_j}{\mathrm{d}z} &= -\mathrm{i}\nu f_j/\varepsilon[e(-\omega_j) + e(\omega_j)\Gamma_j^2] + g_j\,[e(-\omega_j) - e(\omega_j)\Gamma_j^2] \\
\frac{\mathrm{d}\Theta_j}{\mathrm{d}z} &= -\mathrm{i}\nu f_j/\varepsilon e(\omega_j)\Theta_j\Gamma_j - g_j\,e(\omega_j)\Theta_j\Gamma_j \\
\frac{\mathrm{d}T_j}{\mathrm{d}z} &= -2f_j\,\nu/(\varepsilon\omega_j).
\end{aligned}$$

with $J(L) = 1$ and $\Psi(L) = 1$. In view of (A.6) we find, for $\omega_1 \neq \omega_2$, that the drift operator B is *diagonal* and that

$$\frac{\mathrm{d}\langle J\rangle}{\mathrm{d}z} = [f_1^2 + f_2^2 + (f_1 + f_2)^2]\, l\langle J\rangle$$
$$\frac{\mathrm{d}\langle\Psi\rangle}{\mathrm{d}z} = (f_1 + f_2)^2 l\langle\Psi\rangle$$

from which we can conclude (A.3). Finally, note that the above gives the transmitted pulse at depth $z = L$. We obtain the pulse at depth $z = \bar{z}$, the depth at the point of observation, by equating to zero the source terms in the governing equation for Θ in the interval $(\bar{z}, L)$.

Appendix B. The slowness mapping

In this appendix we construct the slowness mapping that we use in section 7.4. This mapping is a simplification of the one used in [36] for the analysis of the operator transport equations. The rationale for the more general mapping used there is to obtain a formulation of the transport equations where some laterally differentiated coupling terms can be controlled.

Denote the mapping by $\hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)$. The slowness mapping is a function of location. Recall the notation $\hat{g} = g(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z))$ and also $\check{g} = g(\boldsymbol{x}, z, \check{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z)$, where $\check{\kappa}$ is the inverse mapping. The mapping is constructed so that

$$\nabla_\perp S^+(\boldsymbol{x}, z, \boldsymbol{\kappa}) = \nabla_\perp S^-(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}). \tag{B.1}$$

We show its existence in a neighbourhood of

$$\boldsymbol{Y}: \quad (z, \boldsymbol{x}, \boldsymbol{\kappa}) = (0, \mathbf{0}, \overline{\boldsymbol{\kappa}})$$

where $\overline{\boldsymbol{\kappa}}$ is the slowness associated with a geometric optics ray going from the source to the point of observation. In the case that the medium is layered the mapping is the identity. In general, the mapping is the identity for $z \leqslant 0$

Let the phase functions $S^{\pm}$ solve the eiconal equation associated with the deterministic medium

$$\begin{aligned} &(\nabla S^{\pm})^2 = \gamma_1^2 \\ &S^+|_{z=z_s} = \boldsymbol{\kappa}\cdot\boldsymbol{x} \\ &S^-|_{z=z_s} = \boldsymbol{\kappa}\cdot\boldsymbol{x} \\ &S_z^+|_{z=z_s} > 0 \\ &S_z^-|_{z=z_s} < 0 \end{aligned} \tag{B.2}$$

with $\gamma_1(\boldsymbol{x}, z) = \sqrt{\rho_0/K_1(\boldsymbol{x}, z)}$ the slowness associated with the effective or deterministic medium. In order to show the existence we define the function $\Re^7 \mapsto \Re^2$

$$H(\boldsymbol{x}, z, \boldsymbol{\kappa}, \hat{\boldsymbol{\kappa}}) = \nabla_\perp S^+(\boldsymbol{x}, z, \boldsymbol{\kappa}) - \nabla_\perp S^-(\boldsymbol{x}, z, \hat{\boldsymbol{\kappa}}). \tag{B.3}$$

We assume that $S^{\pm} \in C^1$. Observe that by definition

$$|\nabla_{\hat{\kappa}} H|_{\mathcal{Y}} = |\nabla_{\check{\kappa}} H|_{\mathcal{Y}} = I$$

with

$$\mathcal{Y}: \quad (z, \boldsymbol{x}, \boldsymbol{\kappa}, \hat{\boldsymbol{\kappa}}) = (0, \mathbf{0}, \overline{\boldsymbol{\kappa}}, \overline{\boldsymbol{\kappa}}).$$

Since $H \in C^1$ it follows by the implicit function theorem [31] (p 67) that a unique and invertible mapping $\hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z) \in C^1$ exists in a neighbourhood of $\boldsymbol{Y}$. The mapping is defined by

$$H(\boldsymbol{x}, z, \boldsymbol{\kappa}, \hat{\boldsymbol{\kappa}}(\boldsymbol{\kappa}; \boldsymbol{x}, z) = \mathbf{0}$$

and (B.1) is thus satisfied in a neighbourhood of $\boldsymbol{Y}$.

Appendix C. Comparison with the theory for small lateral variations

In [26] a theory is developed for a locally layered medium when the lateral variations are weak. We specialize the locally layered medium considered here to this case and check that the approximation given in section 6.1 conforms with that presented in [26].

The governing equations in [26] are given by (3.1). However, the bulk modulus is modelled as

$$K_\varepsilon^{-1}(\boldsymbol{x}, z) = \begin{cases} K_1^{-1} & z \in [0, \infty) \\ K_1^{-1}(1 + \nu(z/\varepsilon^2)) + \varepsilon K_{11}^{-1}(\boldsymbol{x}, z) & z \in (-L, 0) \\ K_2^{-1} & z \in (-\infty, -L]. \end{cases}$$

In this formulation the orientation of the z coordinate has changed and a small deterministic variation has been added to the layered fine-scale fluctuations. Note also that the source is located at the origin. The resulting approximation for the transmitted pressure pulse is found from (8.12), (11.10) and (11.12) of [26],

$$E[p(\boldsymbol{x}, -L, t)|\beta_1(0)] \sim \frac{-\varepsilon^{-3/2}}{2(2\pi)^3} \int\!\!\int\!\!\int \frac{2\zeta_2}{\zeta_1 + \zeta_2} \hat{f}(\omega) \exp[\mathrm{i}\omega[S(\boldsymbol{\kappa}, \boldsymbol{x}, -L) - t]/\varepsilon] \times \exp\big[\mathrm{i}\omega\sqrt{\alpha_{nn}}\beta_1(0) - \omega^2\alpha_{nn}L\big]\, \omega^2 \,\mathrm{d}\omega\, \mathrm{d}\boldsymbol{\kappa} \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{C.1}$$

with

$$S(\boldsymbol{\kappa}, \boldsymbol{x}, -L) = \boldsymbol{\kappa} \cdot \boldsymbol{x} + L\sqrt{\gamma_1^2 - \kappa^2} + \tfrac{1}{2}\varepsilon\zeta_1 \int_{-L}^{0} K_{11}^{-1}\left(\boldsymbol{x} - \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}(L + \sigma), \sigma\right) \mathrm{d}\sigma$$

$$\gamma_i = \sqrt{\rho_0/K_i}$$

$$\zeta_i = \rho_0/\sqrt{\gamma_i^2 - \kappa^2}.$$

In (C.1) we have corrected two misprints in [26]. The factor multiplying $\beta_1(0)$ is corrected to $\sqrt{\alpha_{nn}}$ and we include an ω^2 factor that was left out. It can be shown that (C.1) equals (6.1) to leading order. In (C.1) the term

$$\exp\big[\mathrm{i}\omega\sqrt{\alpha_{nn}}\beta_1(0) - \omega^2\alpha_{nn}L\big] \tag{C.2}$$

represents the effects of the random modulation. Both β_1 and α_{nn} depend on $\boldsymbol{\kappa}$. The integral over the slowness vector $\boldsymbol{\kappa}$ can be evaluated by the stationary phase approximation. Let $\overline{\boldsymbol{\kappa}}$ denote the stationary slowness vector. Using (2.19), (6.4) and (11.4) of [26] we find that

$$\sqrt{\alpha_{nn}}\beta_1(0)|_{\overline{\boldsymbol{\kappa}}} = \chi_\varepsilon/\varepsilon$$

$$\alpha_{nn}L|_{\overline{\boldsymbol{\kappa}}} = (l/2)\int_\Gamma (\gamma_1^*)^2 \cos(\theta^*)^{-1}\, \mathrm{d}u.$$

Thus, both the random travel time correction and the variance of the pulse-shaping function coincide for the two approximations, to leading order.

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