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ON A MODEL FOR A SLIDING DROPLET: WELL-POSEDNESS AND STABILITY OF TRANSLATING CIRCULAR SOLUTIONS*

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Abstract. In this paper the model for a highly viscous droplet sliding down an inclined plane is analyzed. It is shown that, provided the slope is not too steep, the corresponding moving boundary problem possesses classical solutions. Well-posedness is lost when the relevant linearization ceases to be parabolic. This occurs above a critical incline which depends on the shape of the initial wetted region as well as on the liquid's mass. It is also shown that translating circular solutions are asymptotically stable if the kinematic boundary condition is given by an affine function and provided the incline is small.

Key words. sliding droplet, contact angle motion, moving boundary problem, translating solutions

AMS subject classifications. 35R37, 35B40, 35C07, 35Q35

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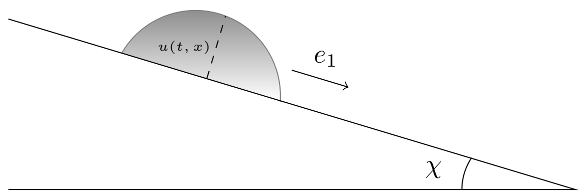
1. Introduction.

Of interest is the analysis of a model for the motion under gravity of a highly viscous droplet on an inclined homogeneous substrate as depicted below.



The droplet is characterized by the wetted region $\Omega(t) = [u(t,\cdot) > 0]$ on the substrate and a height field $u(t,x)$ measured in direction normal to the plane of motion at points $x \in \Omega(t)$, as depicted above. It follows that $u(t,x) = 0$ for $x \in \Gamma(t) := \partial\Omega(t)$. The system reads

$$-\Delta u = \mu x^1 + \lambda \quad \text{in } \Omega(t), \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma(t), \tag{1.2}$$

$$\int_{\Omega(t)} u\, dx = \mathcal{V}, \tag{1.3}$$

$$V = F(-\partial_\nu u) \quad \text{on } \Gamma(t), \tag{1.4}$$

$$\Omega(0) = \Omega^0, \tag{1.5}$$

where the parameter $\mu \geq 0$ is related to the inclination $\chi \geq 0$ via $\mu = \mathrm{Bo}\sin\chi$, where Bo is the Bond number representing the relative magnitude of gravity and viscous

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forces. The unknown λ can be thought of as a time-dependent Lagrange multiplier for the third condition enforcing conservation of the total volume $\mathcal{V} > 0$ (actually, it is a constant resulting from integrating the original fourth-order equation for u; see [2]). The vector ν is the outward unit normal to the boundary $\Gamma(t)$ of the wetted region and V is the speed in normal direction of the same boundary. Coordinates $x = (x^1, x^2) \in \mathbb{R}^2$ are used to describe points on the inclined plane, where x^1 is the coordinate in the direction of motion. Observe that the location of the origin of the coordinate system is irrelevant as any translation in the direction of e_1 is absorbed by the Lagrange multiplier λ and the problem is invariant under translations in the direction of e_2. Equation (1.4) corresponds to a kinematic boundary condition relating the normal velocity to the (small) dynamic contact angle through an empirical law described by the function F. The dimensionless equations (1.1)–(1.5) for an inclined plane were derived in [2] under the assumption that a lubrication approximation is applicable and the Navier–Stokes equations thus greatly simplify. Actually, the model for a droplet on a horizontal plane with $\chi = 0$ and hence $\mu = 0$ was introduced some time ago in [8]. For numerical experiments and numerical schemes in this case we refer the reader to [13] and [6], respectively. For this case local and global existence results for generalized weak solutions are to be found in [7, 9]. Moreover, short time existence of classical solutions was proved in [3], while in [11] circles were identified as the only equilibria and shown to be locally asymptotically stable.

The situation for an inclined plane with $\chi > 0$ (and hence $\mu > 0$) considered here in a two-dimensional context (and in [15] in one dimension) is somewhat different. Indeed, in [2] it was shown that for an affine[1] function F there is a critical inclination of the substrate below which a translating circular solution to (1.1)–(1.5) exists moving at a constant speed, while such a solution ceases to exist if the incline is increased any further. More precisely, consider a droplet sliding down the substrate at a constant speed $v_0 > 0$, so that the wetted region is of the form $\Omega(t) = \Omega_* + tv_0 e_1$ with a still-to-be-determined geometry Ω_* and normal velocity $V = v_0 e_1 \cdot \nu$, where e_1 is the unit vector in x^1-direction and $u(t, x) = u_0(x - tv_0 e_1)$. Then it is readily seen that (1.1)–(1.4) is equivalent to

$$-\Delta u_0 = \mu x^1 + \lambda \qquad \text{in } \Omega_*, \tag{1.6}$$

$$u_0 = 0 \qquad \text{on } \Gamma_*, \tag{1.7}$$

$$\int_{\Omega_*} u_0 \, dx = \mathcal{V}, \tag{1.8}$$

$$F(-\partial_\nu u_0) = v_0 e_1 \cdot \nu \qquad \text{on } \Gamma_* \tag{1.9}$$

for the unknowns u_0, λ, and Ω_* with $\Gamma_* = \partial\Omega_*$. Using polar coordinates, set

$$u_0(r, \theta) := \frac{\mu r}{8}(R_0^2 - r^2)\cos\theta + \frac{2\mathcal{V}}{\pi R_0^4}(R_0^2 - r^2), \quad r \in [0, R_0], \quad \theta \in [0, 2\pi], \tag{1.10}$$

which is easily checked to be the (unique) solution to (1.6)–(1.8) on the disk $\Omega_* = R_0\mathbb{B}$. Here $\mathbb{B} = \mathbb{B}_{\mathbb{R}^2}$ denotes the two-dimensional unit ball centered at the origin. Moreover, (1.9) becomes

$$F\left(\frac{\mu R_0^2}{4}\cos\theta + \frac{4\mathcal{V}}{\pi R_0^3}\right) = v_0 \cos\theta, \quad \theta \in [0, 2\pi]. \tag{1.11}$$

[1]This means that the normal velocity is proportional to the difference between equilibrium and dynamic contact angle for the fluid-substrate system. Such a form was derived in [2] as the linearization for small contact angles of the "Cox–Voinov law," a simple particular choice of the many available laws for F; see [17, 2].

If $\mu = 0$, that is, if the incline vanishes, then (1.11) implies that v_0 must vanish and F must possess a zero at $4\mathcal{V}/\pi R_0^3$. In particular, no translating solution can exist on a disk if $\mu = 0$. If $\mu > 0$, then (1.11) implies that translating solutions only exist provided that F is affine, i.e.,

$$F(q) = aq - b\,, \quad q \in \mathbb{R}\,, \quad a, b > 0\,, \tag{1.12}$$

as in [2]. Conversely, if F satisfies (1.12), then there are a unique radius and a unique velocity, given by

$$R_0 = \left(\frac{4\mathcal{V}a}{\pi b}\right)^{1/3}, \qquad v_0 = \frac{\mu a}{4}\left(\frac{4\mathcal{V}a}{\pi b}\right)^{2/3}, \tag{1.13}$$

such that (1.11) holds. While the velocity is of order μ, the radius is independent of the incline. Consequently, if $\mu > 0$, then (1.12) is a sufficient, but also necessary, condition for a disk to be a translating geometry solution of (1.1)–(1.5), that is, for the existence of a (unique) solution u_0 to problem (1.6)–(1.9) on a disk $\Omega_* = R_0\mathbb{B}$ which is then given by (1.10) with uniquely determined R_0 and v_0 in dependence of a and b. However, to guarantee the positivity of u_0 in Ω_* we need $16\mathcal{V} \geq \mu\pi R_0^5$ as is seen by taking $\theta = \pi$ in (1.10). Writing $\mu = \mathrm{Bo}\sin\chi$ we derive the physical restriction on the maximal inclination of the substrate as

$$\sin\chi \leq \frac{16\mathcal{V}}{\pi \mathrm{Bo} R_0^5}.$$

Therefore, circular solutions to (1.1)–(1.4) only exist if either $\mu = 0$ and F has a zero or $\mu > 0$ is sufficiently small and F is an affine function. It is an interesting question to determine whether noncircular translating solutions exist for general functions F. Observe that this cannot be the case for $\mu = 0$ (and hence $v_0 = 0$ as pointed out above) by Serrin's rigidity theorem [19]. For $\mu > 0$, however, the assumptions of Serrin's theorem are not met since the right-hand sides of (1.6) and (1.9) are no longer constant. Uniqueness of (noncircular) translating solutions is therefore not clear.

In this paper we establish the local well-posedness of (1.1)–(1.5) for general initial droplet geometries $\partial\Omega_0$ and for general laws F for inclines χ smaller than a positive critical value, that is, we do not impose any structural conditions on F except that

$$F \in C^4(\mathbb{R}, \mathbb{R}), \quad F' > 0. \tag{1.14}$$

In contrast to the often used Hanzawa transformation (e.g., see [4, 3, 11] and the references therein) we shall use a description of the unknown curve $\Gamma(t)$ by means of coordinates induced by a smooth flow transversal to the curve $\Gamma_0 = \partial\Omega_0$ (see section 2). This approach to moving boundary problems was first introduced in [10] in a more general context, and the analysis performed in this paper therefore provides a concrete demonstration of the benefits that it offers. In particular, it yields a significant simplification of the analysis required. In short, problem (1.1)–(1.5) is reduced to a single nonlocal, nonlinear evolution equation and yields a relatively simple and insightful formula for its linearization (see Theorem 3.7). The latter is the basis not only for using maximal regularity results to obtain local well-posedness (see Theorem 3.8), but also for the characterization of the critical incline beyond which the model ceases to be parabolic in nature and becomes ill-posed. Finally, we also investigate the stability of the sliding circular droplet, the existence of which is ensured by (1.12). We show that, for small inclines and when starting out with an initial droplet geometry sufficiently close to the circle of radius R_0, the droplet asymptotically becomes circular of radius R_0 sliding down the plane with constant speed v_0 (see Theorem 4.5).

2. Reformulation. System (1.1)–(1.5) can be reduced to a nonlocal geometric evolution for the unknown closed curve $\Gamma(t)$ enclosing the simply connected domain $\Omega(t)$. The derivation of a suitable description of this evolution is the purpose of this section.

Working in a classical regularity context, we consider domains Ω with boundary $\partial\Omega$ of class $buc^{2+\alpha}$ with $\alpha \in (0,1)$, i.e., domains the boundary of which are locally the graph of functions belonging to the little Hölder space $buc^{2+\alpha}$. Recall that, for an open subset O of $\mathbb{R}^n$, the space $\mathrm{BUC}^{2+\alpha}(O)$, defined by

$$\mathrm{BUC}^{2+\alpha}(O) := \big\{u : O \to \mathbb{R} \,\big|\, \partial^\beta u \text{ is uniformly continuous and bounded for } |\beta| \leq 2 \\ \text{and } [\partial^\beta u]_\alpha < \infty \text{ for } |\beta| = 2\big\},$$

is a Banach space with respect to the norm

$$\|u\|_{2,\alpha} := \|u\|_{2,\infty} + \max_{|\beta|=2} [\partial^\beta u]_\alpha, \qquad u \in \mathrm{BUC}^{2+\alpha}(O),$$

for

$$\|u\|_{2,\infty} = \sum_{|\gamma|\leq 2} \|\partial^\gamma u\|_\infty \quad \text{ and } \quad [u]_\alpha := \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Then $buc^{2+\alpha}(O)$ is the closure of $\mathrm{BUC}^{2+\alpha+\epsilon}(O)$ in $\mathrm{BUC}^{2+\alpha}(O)$ for any $\epsilon > 0$. Given any smooth closed curve $\Gamma \in buc^{2+\alpha}$, the space $buc^{2+\alpha}(\Gamma)$ can be defined in the standard way via local charts and a partition of unity.

In the rest of the paper we use the notation $x \cdot y$ and $(x|y)$ interchangeably for the scalar product of vectors $x, y \in \mathbb{R}^2$ to enhance the readability of some formulae.

For a fixed domain Ω with boundary $\Gamma \in buc^{2+\alpha}$ we first solve the subproblem (1.1)–(1.3) to obtain a solution (u, λ). We will use the notation $u^{(f)}$ to indicate the solution of $-\Delta u = f$ in Ω which vanishes on the boundary $\Gamma = \partial\Omega$.

PROPOSITION 2.1. *For any domain Ω bounded by a closed curve $\Gamma \in buc^{2+\alpha}$ and any $\mu \geq 0$, problem* (1.1)–(1.3) *has a unique solution*

$$(u, \lambda) \in buc^{2+\alpha}(\Omega) \times \mathbb{R}$$

with

$$\lambda = \frac{\mathcal{V}}{\int_\Omega u^{(1)}\, dx} - \mu \frac{\int_\Omega u^{(x^1)}\, dx}{\int_\Omega u^{(1)}\, dx}. \tag{2.1}$$

There is $\mu_0 > 0$ depending on Ω and $\mathcal{V}$ such that $u > 0$ in Ω and $-\partial_\nu u > 0$ on Γ if $\mu \in [0, \mu_0)$. The latter does not hold in general. Indeed, if any of the following two conditions:

(i) *The domain Ω is symmetric with respect to the x_2-axis, i.e., it satisfies (after a suitable translation)*

$$(-x^1, x^2) \in \Omega \iff (x^1, x^2) \in \Omega\,,$$

(ii) $\int_\Omega x^1\, dx < 0$ *for a coordinate system in which* $\int_\Omega u^{(x^1)}\, dx = 0$,

are satisfied, then $-\partial_\nu u < 0$ on at least an open subset of Γ if $\mu \geq \mu_1$ for some $\mu_1 > \mu_0$ sufficiently large.

Proof. Since the right-hand side of (1.1) is smooth, classical theory for elliptic boundary value problems [5, Theorem 6.14] ensures that (1.1)–(1.2) has, for any fixed λ, a unique solution $u(\lambda) \in buc^{2+\alpha}(\Omega)$ and

$$u(\lambda) = \mu u^{(x^1)} + \lambda u^{(1)}$$

by linearity of the equation. The unknown parameter λ is then determined by (1.3) from

$$\underbrace{\mu \int_\Omega u^{(x^1)}\, dx}_{=:I_1} + \underbrace{\lambda \int_\Omega u^{(1)}\, dx}_{=:I_0} = \mathcal{V},$$

which yields formula (2.1) and that

$$u = \frac{\mathcal{V}}{I_0} u^{(1)} + \mu \Big[u^{(x^1)} - \frac{I_1}{I_0} u^{(1)} \Big].$$

Assume that

$$I_1 = \int_\Omega u^{(x^1)}\, dx = 0,$$

then $u = \frac{\mathcal{V}}{I_0} u^{(1)} + \mu u^{(x^1)}$. Now, since $u^{(1)} > 0$ in Ω by the maximum principle and since $\partial_\nu u^{(1)} < 0$ on Γ by Hopf's lemma, $\mu_0 > 0$ can be found such that

$$\partial_\nu u = \frac{\mathcal{V}}{I_0} \partial_\nu u^{(1)} + \mu \partial_\nu u^{(x^1)} < 0$$

for $\mu \in [0, \mu_0)$. Now assume that condition (i) holds, i.e., that (after a suitable translation) the domain is symmetric with respect to the x_2-axis. This together with the unique solvability of the boundary value problem implies that $u^{(x^1)}(-x^1, x^2) = -u^{(x^1)}(x^1, x^2)$ for $(x^1, x^2) \in \Omega$. In this case I_1 indeed vanishes. The maximum principle implies that $u^{(x^1)} > 0$ in $[x^1 > 0] \cap \Omega$ and $u^{(x^1)} < 0$ in $[x^1 < 0] \cap \Omega$. Therefore there is a whole region on which $u^{(x^1)} < 0$ and Hopf's lemma yields that

$$\partial_\nu u^{(x^1)} > 0 \text{ on } \partial\Omega \cap [x^1 < 0].$$

Finally, for any $\delta > 0$, one can find $\mu_1 > \mu_0$ sufficiently large such that $\partial_\nu u > 0$ on $\partial\Omega \cap [x^1 \leq -\delta]$ since

$$\partial_\nu u = \frac{\mathcal{V}}{I_0} \underbrace{\partial_\nu u^{(1)}}_{<0} + \mu \underbrace{\partial_\nu u^{(x^1)}}_{>0}.$$

If condition (ii) is satisfied, using that

$$-\int_{\partial\Omega} \partial_\nu u\, d\sigma_{\partial\Omega} = -\int_\Omega \Delta u\, dx = \mu \int_\Omega x^1\, dx + \frac{\mathcal{V}|\Omega|}{I_0} < 0$$

for $\mu \geq \mu_1$ sufficiently large, it easily follows that $\partial_\nu u > 0$ in an open subset of $\partial\Omega$ thanks to the regularity of the solution. ∎

It is not difficult to see that if Ω is a small perturbation of a domain satisfying condition (i) or (ii) in Proposition 2.1, one also has that $-\partial_\nu u < 0$ on at least an open subset of Γ if $\mu \geq \mu_1$.

With u in hand, equations (1.4)–(1.5) amount to a nonlocal (geometric) evolution for the closed curve $\Gamma(t)$. In order to obtain an evolution equation for it, it is necessary to gain a local understanding of the manifold $\mathcal{M}^{2+\alpha}$ of closed curves in $\mathbb{R}^2$ of class $buc^{2+\alpha}$. A convenient local parametrization about a given fixed initial curve Γ_0 is particularly useful. Let τ_0, ν_0 be the unit tangent and normal vectors for Γ_0, the latter pointing out of the domain enclosed by Γ_0.

LEMMA 2.2 (tubular neighborhood). *Given $\Gamma_0 \in \mathcal{M}^{2+\alpha}$, there is $r_0 > 0$ such that*

$$T_{r_0}(\Gamma_0) := \{x \in \mathbb{R}^2 \,|\, |d(x, \Gamma_0)| < r_0\}$$

is an open neighborhood of Γ_0 diffeomorphic to $\Gamma_0 \times (-r_0, r_0)$. The notation $d(\cdot, \Gamma_0)$ is used for the signed distance function to Γ_0.

Proof. Define the map

$$\Phi : \Gamma_0 \times (-r_0, r_0) \longrightarrow \mathbb{R}^2,\ (y, r) \mapsto y + r\nu_0(y),$$

and recall that, if $\gamma_0 = \gamma_0(s)$ is an arc length parametrization of Γ_0 about the point $y \in \Gamma_0$, then

$$\gamma_0'(s) = \tau_0(\gamma_0(s)) \quad \text{and} \quad d_s\nu_0(\gamma_0(s)) = \kappa_0(\gamma_0(s))\tau_0(\gamma_0(s)),$$

where κ_0 denotes the curvature of Γ_0. Hence, computing in the above coordinates,

$$d_y\Phi(y, r) = \tau_0(y) + r\kappa_0(y)\tau_0(y) \quad \text{and} \quad d_r\Phi(y, r) = \nu_0(y).$$

The assumption on Γ_0 implies that

$$\|\kappa_0\|_\infty < \infty,$$

and thus that $d\Phi(y, r)$ is invertible at least as long as $1 + r\kappa_0(y) > 0$, since its columns are orthogonal. This can be ensured by choosing $r_0 < 1/\|\kappa_0\|_\infty$ and the inverse function theorem yields local invertibility of Φ. Assuming without loss of generality that r_0 is also so small that

$$\mathbb{B}_{\Gamma_0}(y, 2r_0) \times (-2r_0, 2r_0) \cap \Gamma_0 = \mathbb{B}_{\Gamma_0}(y, 2r_0),$$

for each $y \in \Gamma_0$, it follows that $x \in T_{r_0}(\Gamma_0)$ has a unique representation by Φ, yielding global injectivity. $\square$

Remark 2.3. The map Φ yields a foliation of $T_{r_0}(\Gamma_0)$ by $buc^{1+\alpha}$ curves since it uses $\nu_0 \in buc^{1+\alpha}(\Gamma_0)$. For technical reasons this regularity is not sufficient.

LEMMA 2.4 (generalized tubular neighborhood). *Given $\Gamma_0 \in \mathcal{M}^{2+\alpha}$, there is $r_0 > 0$ and curves $\Gamma_r \in \mathcal{M}^{2+\alpha}$ for $r \in (-r_0, r_0)$ such that*

$$\bigcup_{|r|<r_0} \Gamma_r$$

is an open neighborhood of Γ_0 that is diffeomorphic to $\Gamma_0 \times (-r_0, r_0)$.

Proof. By the preceding lemma, there exists $\tilde{r}_0$ such that, for each $x \in T_{\tilde{r}_0}(\Gamma_0)$, there is a unique $(y, r) = \big(y(x), r(x)\big) \in \Gamma_0 \times (-\tilde{r}_0, \tilde{r}_0)$ with $x = y + r\nu_0(y)$. In $T_{\tilde{r}_0}(\Gamma_0)$ define the field

$$\tilde{\tilde{\nu}}(x) := \nu_0\big(y(x)\big), \quad x \in T_{\tilde{r}_0}(\Gamma_0),$$

take a smooth cutoff function $\eta : \mathbb{R} \to \mathbb{R}$ with

$$0 \le \eta \le 1, \quad \eta\big|_{[-\tilde{r}_0/2, \tilde{r}_0/2]} \equiv 1, \quad \text{and} \quad \eta\big|_{(-3\tilde{r}_0/4, 3\tilde{r}_0/4)^c} \equiv 0,$$

and set

$$\tilde{\nu}(x) := \begin{cases} \tilde{\tilde{\nu}}(x)\eta\big(r(x)\big), & x \in T_{\tilde{r}_0}(\Gamma_0), \\ 0, & x \notin T_{\tilde{r}_0}(\Gamma_0). \end{cases}$$

Then $\tilde{\nu} \in buc^{1+\alpha}(\mathbb{R}^2, \mathbb{R}^2)$ is a global vector field. Now take a compactly supported smooth mollifier ψ_δ and define

$$\nu^\delta := \psi_\delta * \tilde{\nu}$$

componentwise. It follows that $\nu^\delta \in \mathrm{BUC}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and that

$$\nu^\delta \longrightarrow \nu_0 \circ y \;\text{ in }\; buc^{1+\alpha}\big(T_{\tilde{r}_0/2}(\Gamma_0)\big) \tag{2.2}$$

as $\delta \to 0$. In particular it holds that

$$|\nu^\delta(y) \cdot \tau_0(y)| \le c(\delta) < 1, \;\; y \in \Gamma_0,$$

where $c(\delta) \to 0$ as $\delta \to 0$. The vector field is therefore uniformly transversal to Γ_0 for $\delta > 0$ fixed small enough. Then let $\varphi^\delta = \varphi^\delta(y, r)$ be the flow generated by the ODE

$$\begin{cases} \dot{x} = \frac{d}{dr}x = \nu^\delta(x), \\ x(0) = y \in \Gamma_0. \end{cases}$$

It is easily seen that there is $r_0 > 0$ such that φ^δ is defined on $\Gamma_0 \times (-r_0, r_0)$ with

$$\Gamma_r := \varphi^\delta(\Gamma_0, r) \subset T_{\tilde{r}_0}(\Gamma_0), \;\; |r| < r_0$$

and standard ODE arguments yield that

$$\varphi^\delta : \Gamma_0 \times (-r_0, r_0) \longrightarrow \bigcup_{|r|<r_0} \Gamma_r$$

is a diffeomorphism of class $buc^{2+\alpha}$. □

In the following we use the notation introduced in the proof of Lemma 2.4 for a fixed $\Gamma_0 \in \mathcal{M}^{2+\alpha}$, that is, having $\delta > 0$ chosen small enough we let

$$\varphi^\delta : \Gamma_0 \times (-r_0, r_0) \to \mathbb{R}^2$$

denote the flow induced by the vector field $\nu^\delta \in \mathrm{BUC}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ transversal to Γ_0. It is convenient to define

$$\varphi_r^\delta(y) := \varphi^\delta(y, r), \qquad \nu_r^\delta(y) := \nu^\delta\big(\varphi_r^\delta(y)\big)$$

for $(y, r) \in \Gamma_0 \times (-r_0, r_0)$, hence

$$\frac{d}{dr}\varphi_r^\delta(y) = \nu_r^\delta(y), \quad (y, r) \in \Gamma_0 \times (-r_0, r_0). \tag{2.3}$$

Note that $\nu_0^\delta = \nu^\delta$ since $\varphi_0^\delta = \mathrm{id}$. The previous lemma provides coordinates (y, r) for a neighborhood of Γ_0, which can be denoted by $T_{r_0}^{\nu^\delta}(\Gamma_0)$ since it is constructed based on ν^δ. Explicitly this means

$$\forall\, x \in T_{r_0}^{\nu^\delta}(\Gamma_0)\; \exists!\, (y, r) \in \Gamma_0 \times (-r_0, r_0) \text{ such that } x = \varphi_r^\delta(y). \tag{2.4}$$

We show next that curves close to Γ_0 can be conveniently parametrized in these coordinates.

LEMMA 2.5. *Let Γ_0 and $\Gamma \in \mathcal{M}^{2+\alpha}$ be close in the* BUC^1*-topology, that is, let them satisfy*

$$d_H\Big(\big\{(\tilde{y}, \nu_\Gamma(\tilde{y}))\,\big|\,\tilde{y} \in \Gamma\big\}, \big\{\big(y, \nu_0(y)\big)\,\big|\,y \in \Gamma_0\big\}\Big) \ll 1,$$

where d_H denotes the Hausdorff distance between compact sets. Then there is a unique function $\rho \in buc^{2+\alpha}(\Gamma_0)$ such that

$$\Gamma = \big\{\varphi^\delta\big(y, \rho(y)\big)\,\big|\,y \in \Gamma_0\big\} = \varphi^\delta_\rho(\Gamma_0).$$

Proof. We refer the reader to [10, Lemma 2.6] for a complete proof. ∎

3. The equation for $\Gamma(t)$ and its linearization. We now focus on equations (1.4)–(1.5) for a given simply connected domain Ω^0 of class $buc^{2+\alpha}$ with $\alpha \in (0,1)$, that is, $\Gamma_0 := \partial\Omega^0 \in \mathcal{M}^{2+\alpha}$. Using the corresponding coordinates introduced in (2.4) with $\delta > 0$ small enough, it is possible to reduce the evolution for $\Gamma(t)$ to one for the function $\rho(t,\cdot) : \Gamma_0 \to \mathbb{R}$ given in Lemma 2.5 through

$$\Gamma(t) = \big\{\varphi^\delta_{\rho(t,y)}(y)\,\big|\,y \in \Gamma_0\big\} = \varphi^\delta_{\rho(t,\cdot)}(\Gamma_0) =: \Gamma_{\rho(t)}.$$

Denote the unit tangent and normal vectors to $\Gamma_{\rho(t)}$ at the point $\varphi^\delta_{\rho(t,y)}(y)$ by $\tau_{\rho(t)}(y)$ and $\nu_{\rho(t)}(y)$, respectively. Then (2.3) implies

$$\frac{d}{dt}\varphi^\delta_{\rho(t,y)}(y) = \nu^\delta_{\rho(t,y)}(y)\dot{\rho}(t,y), \quad y \in \Gamma_0,$$

with superscript dot indicating a derivative with respect to time. Since the normal velocity $V_{\Gamma_{\rho(t)}}$ of $\Gamma_{\rho(t)}$ at a point $\varphi^\delta_{\rho(t,y)}(y)$ is given by the component of the projection of the tangent vector to the curve $t \mapsto \varphi^\delta_{\rho(t,y)}(y)$ onto the unit normal vector $\nu_{\rho(t)}(y)$ at that point, it holds that

$$V_{\Gamma_{\rho(t)}} = \big(\nu^\delta_{\rho(t,y)}(y)\big|\nu_{\rho(t)}(y)\big)\dot{\rho}(t,y) \quad \text{for} \quad \varphi^\delta_{\rho(t,y)}(y) \in \Gamma(t) \text{ with } y \in \Gamma_0.$$

To keep notation simple we often omit the time variable. Therefore, (1.4)–(1.5) is equivalent to the evolution equation for ρ,

$$\begin{cases} \dot{\rho} = \big(\nu^\delta_\rho\big|\nu_\rho\big)^{-1}F\big(-\partial_{\nu_\rho}u_\rho\big) =: G(\rho), \quad t > 0, \\ \rho(0,\cdot) \equiv 0, \end{cases} \tag{3.1}$$

where (u_ρ, λ_ρ) denotes the solution of (1.1)–(1.3) from Proposition 2.1 in Ω_ρ, the domain bounded by Γ_ρ, and a given fixed $\mu \geq 0$ (note that $\rho \equiv 0$ gives the initial domain, that is, $\Omega_0 = \Omega^0$). This is a nonlinear, nonlocal equation for $\rho : [0,\infty) \times \Gamma_0 \to \mathbb{R}$. Notice that, for $\delta \ll 1$ and $|\rho| \ll 1$, the factor of $\dot{\rho}$ in the expression for V_{Γ_ρ} satisfies

$$\nu^\delta_\rho \cdot \nu_\rho \simeq \nu_0 \cdot \nu_0 = 1.$$

In order to obtain local well-posedness for (3.1) using maximal regularity techniques, its linearization at the initial datum $\rho_0 \equiv 0$ is computed. For this we first note the following proposition.

PROPOSITION 3.1. *There exists an open zero-neighborhood $\mathcal{O}$ in $buc^{2+\alpha}(\Gamma_0)$ such that*

$$G \in C^2(\mathcal{O}, buc^{1+\alpha}(\Gamma_0)).$$

Proof. This follows from (1.14) and the fact that the flow $\varphi^\delta_r = \varphi^\delta(\cdot, r)$ is smooth with respect to r, which implies that the solution (u_ρ, λ_ρ) of (1.1)–(1.3) depends smoothly on ρ since $\Gamma_\rho = \varphi^\delta_\rho(\Gamma_0)$. We refer the reader to [10, Theorem 3.6] for a more detailed and explicit calculation of this derivative which yields the desired smoothness. ∎

From Proposition 3.1 it follows that

$$DG(0) \in \mathcal{L}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)\,,$$

where $\mathcal{L}$ stands for the bounded linear operators. We next verify that $-DG(0)$ is the generator of an analytic semigroup (if μ is sufficiently small). Observe that

$$\begin{aligned} DG(0)h &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} G(\epsilon h) \\ &= -\frac{1}{\nu^\delta\cdot\nu_0}F'(-\partial_{\nu_0}u_0)\Big\{\frac{d}{d\epsilon}\Big|_{\epsilon=0}\partial_{\nu_{\epsilon h}}u_{\epsilon h}\Big\} \\ &\quad -\frac{1}{(\nu^\delta\cdot\nu_0)^2}F(-\partial_{\nu_0}u_0)\Big\{\frac{d}{d\epsilon}\Big|_{\epsilon=0}\big(\nu^\delta_{\epsilon h}\cdot\nu_{\epsilon h}\big)\Big\} \end{aligned} \tag{3.2}$$

for $h \in buc^{2+\alpha}(\Gamma_0)$ since $\nu^\delta_0 = \nu^\delta$. It then remains to compute the derivatives in the curly brackets. This is where the choice of coordinate system from Lemma 2.4 delivers its benefits yielding a particularly insightful representation.

LEMMA 3.2. *It holds that*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\nu_{\epsilon h} = -\Big[h\big(d_y\nu^\delta[\tau_0]\big|\nu_0\big) + h'\big(\nu^\delta\big|\nu_0\big)\Big]\tau_0$$

for $h \in buc^{2+\alpha}(\Gamma_0)$, where $'$ denotes differentiation with respect to arc length.

Proof. It follows from $\tau_\rho\cdot\nu_\rho \equiv 0$ and $\nu_\rho\cdot\nu_\rho \equiv 1$ that

$$\Big(\frac{d}{d\epsilon}\tau_{\epsilon h}\Big)\cdot\nu_{\epsilon h} = -\tau_{\epsilon h}\cdot\Big(\frac{d}{d\epsilon}\nu_{\epsilon h}\Big), \quad \Big(\frac{d}{d\epsilon}\nu_{\epsilon h}\Big)\cdot\nu_{\epsilon h} = 0.$$

This implies that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\nu_{\epsilon h} = -\Big[\Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tau_{\epsilon h}\Big)\cdot\nu_0\Big]\tau_0. \tag{3.3}$$

It is therefore enough to compute $\frac{d}{d\epsilon}\big|_{\epsilon=0}\tau_{\epsilon h}$. Now, since $\gamma_\rho(s) := \varphi^\delta\big(\gamma_0(s), \rho\big(\gamma_0(s)\big)\big)$ is a parametrization of Γ_ρ whenever γ_0 is an arc length parametrization of Γ_0, one has from (2.3) that

$$\tilde\tau_{\epsilon h} := d_y\varphi^\delta(\cdot, \epsilon h)[\tau_0] + \epsilon h'\nu^\delta_{\epsilon h}$$

is a tangent vector to $\Gamma_{\epsilon h}$ and thus that $\tau_{\epsilon h} = \tilde\tau_{\epsilon h}/|\tilde\tau_{\epsilon h}|$. The latter yields

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0}\tau_{\epsilon h} &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}\frac{\tilde\tau_{\epsilon h}}{|\tilde\tau_{\epsilon h}|} = \frac{1}{|\tilde\tau_0|}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde\tau_{\epsilon h} - \frac{1}{|\tilde\tau_0|^3}\Big[\tilde\tau_0\cdot\Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde\tau_{\epsilon h}\Big)\Big]\tilde\tau_0 \\ &= \frac{1}{|\tilde\tau_0|}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde\tau_{\epsilon h} - \frac{1}{|\tilde\tau_0|}\Big[\tau_0\cdot\Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde\tau_{\epsilon h}\Big)\Big]\tau_0. \end{aligned} \tag{3.4}$$

Then, using (2.3) again,

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \tilde{\tau}_{\epsilon h} &= d_y\Big(\frac{d}{dr}\varphi^\delta(\cdot,0)\Big)[\tau_0]h + h'\nu_0^\delta \\ &= h d_y \nu_0^\delta[\tau_0] + h'\nu_0^\delta. \end{aligned} \tag{3.5}$$

Combining (3.3)–(3.5) and noticing $\tilde{\tau}_0 = d_y\varphi^\delta(\cdot,0)[\tau_0] = \tau_0$ since $\varphi_0^\delta = \mathrm{id}_{\Gamma_0}$, we get

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \nu_{\epsilon h} &= -\Big[\Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} \tilde{\tau}_{\epsilon h}\Big)\cdot \nu_0\Big]\tau_0 \\ &= -\big[h\big(d_y\nu_0^\delta[\tau_0]\big|\nu_0\big) + h'\big(\nu_0^\delta\big|\nu_0\big)\big]\tau_0 \end{aligned}$$

and the claim follows from $\nu_0^\delta = \nu^\delta$. □

LEMMA 3.3. *It holds that*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \big(\nu_{\epsilon h}^\delta \cdot \nu_{\epsilon h}\big) = \Big[h\big(d_y\nu^\delta[\nu_0]\big|\nu_0\big) - h'\big(\nu^\delta\big|\tau_0\big)\Big]\big(\nu^\delta\big|\nu_0\big)$$

for $h \in buc^{2+\alpha}(\Gamma_0)$.

Proof. Owing to the previous lemma it only remains to compute

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \nu_{\epsilon h}^\delta = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \nu^\delta\big(\varphi^\delta(\cdot,\epsilon h)\big) = h d_y \nu^\delta[\nu^\delta]$$

and to note that

$$\nu^\delta - \big(\nu^\delta\big|\tau_0\big)\tau_0 = \big(\nu^\delta\big|\nu_0\big)\nu_0.$$

The assertion then follows. □

Remark 3.4. Notice that when $\delta \simeq 0$ one has that

$$d_y\nu^\delta[\nu^\delta] \simeq 0, \quad \nu^\delta\cdot\tau_0 \simeq 0, \quad \text{and} \quad \nu_0^\delta\cdot\nu_0 \simeq 1,$$

uniformly on Γ_0. If δ can be set equal to zero, then

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \nu_{\epsilon h}^0 \cdot \nu_{\epsilon h} \equiv 0.$$

LEMMA 3.5. *Given* $h \in buc^{2+\alpha}(\Gamma_0)$ *and* $\epsilon > 0$ *small, let* $u_{\epsilon h}^{(f)}$ *solve the boundary value problem*

$$\begin{cases} -\Delta u_{\epsilon h}^{(f)} = f & \text{in } \Omega_{\epsilon h} = \varphi_{\epsilon h}^\delta(\Omega_0), \\ u_{\epsilon h}^{(f)} = 0 & \text{on } \Gamma_{\epsilon h} = \varphi_{\epsilon h}^\delta(\Gamma_0) \end{cases}$$

for $f \in \mathrm{C}^\infty(\mathbb{R}^n)$. *Then*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_{\nu_{\epsilon h}} u_{\epsilon h}^{(f)} = -DtN_{\Gamma_0}\big([\partial_{\nu^\delta}u_0^{(f)}]h\big) + \big(\nu_0^\mathsf{T} D^2 u_0^{(f)}\nu^\delta\big)h,$$

where the Dirichlet-to-Neumann operator DtN_Γ *is the operator that yields the normal derivative* $\partial_{\nu_\Gamma} w_g$ *(Neumann datum) of the harmonic function* w_g *with Dirichlet datum* g*, that is*

$$DtN_\Gamma(g) = \partial_{\nu_\Gamma} w_g,$$

and where $u_0^{(f)}$ *is the solution corresponding to the boundary value problem in* Ω_0.

Proof. Assume first that $h \le 0$, hence $\Omega_{\epsilon h} \subset \Omega_0$. We look for $u^{(f)}_{\epsilon h}$ in the form

$$u^{(f)}_{\epsilon h} = v_{\epsilon h} + u^{(f)}_0 .$$

Then $v_{\epsilon h}$ satisfies

$$\begin{cases} -\Delta v_{\epsilon h} = 0 & \text{in } \Omega_{\epsilon h}, \\ v_{\epsilon h} = -u^{(f)}_0 \big|_{\Gamma_{\epsilon h}} & \text{on } \Gamma_{\epsilon h}, \end{cases}$$

and

$$\partial_{\nu_{\epsilon h}} u^{(f)}_{\epsilon h} = \partial_{\nu_{\epsilon h}} v_{\epsilon h} + \partial_{\nu_{\epsilon h}} u^{(f)}_0 = -DtN_{\Gamma_{\epsilon h}}(u^{(f)}_0\big|_{\Gamma_{\epsilon h}}) + \nu_{\epsilon h} \cdot \nabla u^{(f)}_0 \big|_{\Gamma_{\epsilon h}} .$$

It is known that the mapping

$$\mathcal{M}^{2+\alpha} \to \mathcal{L}\big(buc^{2+\alpha}(\Gamma), buc^{1+\alpha}(\Gamma)\big), \quad \Gamma \mapsto DtN_\Gamma ,$$

is a smooth local section of the corresponding bundle. Indicating with a superscript $*$ the pull-back operation, it follows that (see [10, section 3] for more details)

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_{\nu_{\epsilon h}} u^{(f)}_{\epsilon h} = & - \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\varphi^\delta_{\epsilon h})^* DtN_{\Gamma_{\epsilon h}} \Big(\big((\varphi^\delta_{\epsilon h})^{-1} \big)^* u^{(f)}_0 \big|_{\Gamma_0} \Big) \\ & - DtN_{\Gamma_0} \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} (\varphi^\delta_{\epsilon h})^* u^{(f)}_0 \big|_{\Gamma_{\epsilon h}} \Big) \\ & + \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} \nu_{\epsilon h} \Big) \cdot \nabla u^{(f)}_0 \big|_{\Gamma_0} \\ & + \nu_0 \cdot \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} (\varphi^\delta_{\epsilon h})^* \nabla u^{(f)}_0 \big|_{\Gamma_{\epsilon h}} \Big) \\ = & - DtN_{\Gamma_0} \Big([\partial_{\nu^\delta} u^{(f)}_0] h \Big) + \Big(\sum_{j,k=1}^{2} \nu^j_0 (\nu^\delta)^k \frac{\partial^2}{\partial x^j \partial x^k} u^{(f)}_0 \Big) h, \end{aligned} \tag{3.6}$$

since the first term after the first equality sign vanishes in view of the homogeneous Dirichlet condition satisfied by $u^{(f)}_0$ and the third in view of Lemma 3.2 and of the boundary condition again. It remains to show that the result remains valid for any h without the restriction that $h \le 0$. To that end, define $\Gamma_{r_0} = \varphi^\delta(\Gamma_0, r_0)$ for $r_0 > 0$ small enough and replace the solution $u^{(f)}_0$ by the solution $u^{(f)}_{r_0}$ in the above argument. At the end of the calculation, formula (3.6) is obtained with all terms after the first equality sign nonvanishing. Letting r_0 tend to zero makes them vanish and delivers the claim. For more details we refer the reader to the proof of [10, Theorem 3.7]. ∎

LEMMA 3.6. *Given $h \in buc^{2+\alpha}(\Gamma_0)$ it holds that*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_{\nu_{\epsilon h}} u_{\epsilon h} = -DtN_{\Gamma_0}\Big([\partial_{\nu^\delta} u_0] h \Big) + \Big(\nu_0^\mathsf{T} D^2 u_0 \nu^\delta \Big) h + (\partial_{\nu_0} u^{(1)}_0) \, \frac{d}{d\epsilon}\Big|_{\epsilon=0} \lambda_{\epsilon h},$$

where

$$\lambda_{\epsilon h} = \frac{\mathcal{V}}{\int_{\Omega_{\epsilon h}} u^{(1)}_{\epsilon h} \, dx} - \mu \frac{\int_{\Omega_{\epsilon h}} u^{(x^1)}_{\epsilon h} \, dx}{\int_{\Omega_{\epsilon h}} u^{(1)}_{\epsilon h} \, dx}. \tag{3.7}$$

Proof. Recall from Proposition 2.1 that

$$u_{\epsilon h} = \mu u^{(x^1)}_{\epsilon h} + \lambda_{\epsilon h} u^{(1)}_{\epsilon h}$$

with $\lambda_{\epsilon h}$ given as in the statement. Lemma 3.5 implies that

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_{\nu_{\epsilon h}} u^{(x^1)}_{\epsilon h} &= -DtN_{\Gamma_0}\Big([\partial_{\nu^\delta} u^{(x^1)}_0]h\Big) + \Big(\nu_0^{\mathsf{T}} D^2 u^{(x^1)}_0 \nu^\delta\Big)h,\\
\frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_{\nu_{\epsilon h}} u^{(1)}_{\epsilon h} &= -DtN_{\Gamma_0}\Big([\partial_{\nu^\delta} u^{(1)}_0]h\Big) + \Big(\nu_0^{\mathsf{T}} D^2 u^{(1)}_0 \nu^\delta\Big)h.
\end{aligned}$$

It follows that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_{\nu_{\epsilon h}} u_{\epsilon h} = -DtN_{\Gamma_0}\Big([\partial_{\nu^\delta} u_0]h\Big) + \Big(\nu_0^{\mathsf{T}} D^2 u_0 \nu^\delta\Big)h + (\partial_{\nu_0} u^{(1)}_0)\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} \lambda_{\epsilon h}. \qquad \square$$

Combining the results of (3.2), Lemma 3.3, and Lemma 3.6, the linearization of G at zero is seen to be given by the expression

$$\begin{aligned}
DG(0)h = &- \frac{1}{\nu^\delta \cdot \nu_0} F'(-\partial_{\nu_0} u_0)\Big\{ - DtN_{\Gamma_0}\Big([\partial_{\nu^\delta} u_0]h\Big)\\
&\qquad\qquad + \Big(\nu_0^{\mathsf{T}} D^2 u_0 \nu^\delta\Big)h + (\partial_{\nu_0} u^{(1)}_0)\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} \lambda_{\epsilon h}\Big\}\\
&- \frac{1}{(\nu^\delta \cdot \nu_0)} F(-\partial_{\nu_0} u_0)\Big\{ h\big(d_y \nu^\delta[\nu_0]\big|\nu_0\big) - h'\big(\nu^\delta\big|\tau_0\big)\Big\}\\
=:&\, I + II + III + IV + V
\end{aligned}$$

for $h \in buc^{2+\alpha}(\Gamma_0)$, where (u_0, λ_0) is the solution to (1.1)–(1.3) in Ω_0 from Proposition 2.1. From this formula we derive the following generation result.

THEOREM 3.7. *Suppose* (1.14) *and let* $\Gamma_0 \in \mathcal{M}^{2+\alpha}$. *Then*

$$-DG(0) \in \mathcal{H}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$$

for $\mu \in [0, \mu_0)$, *where* $\mu_0 > 0$ *is small enough depending on* Ω_0 *and* $\mathcal{V}$. *In other words,* $DG(0)$ *generates an analytic* C^0*-semigroup on* $buc^{1+\alpha}(\Gamma_0)$ *for* $\mu < \mu_0$ *and its domain of definition coincides with* $buc^{2+\alpha}(\Gamma_0)$. *If* Ω_0 *satisfies condition* (i) *or condition* (ii) *of Proposition* 2.1, *then there is* $\mu_1 \geq \mu_0$ *such that* $\partial_t + DG(0)$ *is backward parabolic on an open subset of* Γ *for* $\mu \geq \mu_1$, *which makes the evolution equation linearly ill-posed.*

Proof. First observe that for $\partial_{\nu^\delta} u_0$ we have

$$\begin{aligned}
-\Delta \partial_{\nu^\delta} u_0 &= - \sum_{j,k=1}^{2} \partial_j^2\big((\nu^\delta)^k \partial_k u_0\big)\\
&= - \sum_{j,k=1}^{2} \Big\{ \big(\partial_j^2 (\nu^\delta)^k\big)\partial_k u_0 + 2\partial_j (\nu^\delta)^k \partial_j \partial_k u_0 + (\nu^\delta)^k \partial_j^2 \partial_k u_0 \Big\}\\
&= - \Delta \nu^\delta \cdot \nabla u_0 - 2D\nu^\delta : D^2 u_0 - \nu^\delta \cdot \nabla \Delta u_0\\
&= - \Delta \nu^\delta \cdot \nabla u_0 - 2D\nu^\delta : D^2 u_0 + \mu(\nu^\delta)^1 \in buc^\alpha(\Omega_0)
\end{aligned}$$

by definition of $u_0 \in buc^{2+\alpha}(\Omega_0)$ and

$$\begin{aligned}\partial_{\nu_0}\big(\partial_{\nu^\delta} u_0\big) &= \sum_{k=1}^{2} \nu_0^k \partial_k \Big(\big[(\nu^\delta|\nu_0)\nu_0 + (\nu^\delta|\tau_0)\tau_0\big] \cdot \nabla u_0\Big)\\ &= \big(\partial_{\nu_0}(\nu^\delta|\nu_0)\big)\partial_{\nu_0} u_0 + (\nu^\delta|\nu_0)\partial_{\nu_0\nu_0} u_0\\ &= \big(\partial_{\nu_0}\nu^\delta|\nu_0\big)\partial_{\nu_0} u_0 - (\nu^\delta|\nu_0) f \in buc^{1+\alpha}(\Gamma_0) \qquad (3.8)\end{aligned}$$

for $f(x) := \mu x^1 + \lambda$, where we used for the second equality that $\partial_{\tau_0} u_0 = 0$ owing to the Dirichlet boundary condition and for the third equality the fact that $\partial_{\nu_0}\nu_0 = 0$ along with

$$\partial_{\nu_0\nu_0} u_0 = \partial_{\nu_0\nu_0} u_0 + \partial_{\tau_0\tau_0} u_0 = \Delta u_0 = -f \quad \text{on } \Gamma_0.$$

Consequently, classical theory of boundary value problems [5, Theorem 6.14] implies that

$$\partial_{\nu^\delta} u_0 \in buc^{2+\alpha}(\Gamma_0).$$

From this it follows that

$$\Big[h \mapsto \frac{1}{\nu^\delta \cdot \nu_0} F'(-\partial_{\nu_0} u_0) DtN_{\Gamma_0}\Big([\partial_{\nu^\delta} u_0] h\Big)\Big] \in \mathcal{L}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big).$$

Next notice that the map that associates with a curve Γ the corresponding λ from Proposition 2.1 is well-defined in a neighborhood of Γ_0 in $\mathcal{M}^{2+\alpha}$. Consequently, its tangential map

$$buc^{2+\alpha}(\Gamma_0) \longrightarrow \mathbb{R}, \quad h \mapsto \frac{d}{d\epsilon}\Big|_{\epsilon=0} \lambda_{\epsilon h}$$

is a linear, rank 1, and, hence, compact operator. Now term I is the most important one and defines an elliptic pseudodifferential operator of order 1 whenever

$$-\partial_{\nu^\delta} u_0 > 0,\ F'(-\partial_{\nu_0} u_0) > 0, \ \text{ and } \ \nu^\delta \cdot \nu_0 > 0 \quad \text{on } \ \Gamma_0.$$

The second condition is satisfied by assumption (1.14). The first condition holds true if $-\partial_{\nu_0} u_0 > 0$ on Γ_0, which follows from Proposition 2.1 provided $\mu \in [0, \mu_0)$. Indeed, if δ is small enough, then $\nu^\delta \cdot \nu_0 \simeq 1$ (which also guarantees the validity of the third condition), and therefore we have that

$$-\partial_{\nu^\delta} u_0 \simeq -\partial_{\nu_0} u_0 > 0,$$

thanks to the uniform convergence in $buc^{1+\alpha}$ in (2.2). This implies that I is in fact the generator of an analytic C^0-semigroup on $buc^{1+\alpha}(\Gamma_0)$ with domain $buc^{2+\alpha}(\Gamma_0)$. A complete argument would require a standard localization argument based on the smoothness of the coefficients and a symbol analysis of the corresponding frozen coefficients operator (see, e.g., [4, 3] for more details). In this case, the principal symbol has the explicit form $a_0|\xi|$ because of the particularly insightful form of the main term of $DG(0)$, which, it is reminded, is itself a consequence of the use of coordinates constructed by means of the flow φ^δ. The remaining terms can be treated by perturbation arguments. Indeed, as multiplication operators the terms II–IV are lower order perturbations thanks to regularity of the coefficients (using also (3.8)). The term V is a small perturbation due to

$$\nu^\delta \cdot \tau_0 \simeq 0$$

so that one can use the fact that the set of analytic generators is open in the natural operator topology of $\mathcal{L}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$. The first assertion of Theorem 3.7 is therefore proved. As for the second, we note that Proposition 2.1 yields $\mu_1 \geq \mu_0$ large enough such that $-\partial_{\nu^\delta} u_0 < 0$ on an open subset $\mathcal{U}$ of Γ_0 for $\mu \geq \mu_1$. In this case the local nature of the operator is captured by the symbol $-a_0|\xi|$ on $\mathcal{U}$ which has the wrong sign and points to the backward character of the evolution equation in $\mathcal{U}$. ◻

Existence results based on maximal regularity can now be used to derive the following theorem.

THEOREM 3.8. *Given any $\Gamma_0 \in buc^{2+\alpha}$, there is a $\mu_0 > 0$ (depending on Γ_0 and $\mathcal{V}$) such that, for all $\mu \in [0, \mu_0)$, system* (1.1)–(1.5) *is well-posed on some maximal interval* $[0, T_0)$. *The solution* (u, Γ) *satisfies*

$$\Gamma(t) = \Gamma_{\rho(t,\cdot)}, \quad t \in [0, T_0),$$

with

$$\rho \in \mathrm{C}\big([0, T_0), buc^{2+\alpha}(\Gamma_0)\big) \cap \mathrm{C}^1\big([0, T_0), buc^{1+\alpha}(\Gamma_0)\big),$$

and

$$0 < u(t, \cdot) \in buc^{2+\alpha}\big(\Omega(t)\big), \quad t \in [0, T_0),$$

for $\Omega(t) = \Omega_{\rho(t,\cdot)}$. If Ω_0 satisfies condition (i) *or condition* (ii) *of Proposition* 2.1, *then there is $\mu_1 \geq \mu_0$ such that the system* (1.1)–(1.5) *is linearly ill-posed for $\mu \geq \mu_1$.*

Proof. Since $\mathcal{H}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$ is open in $\mathcal{L}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$, it follows from Theorem 3.7 and Proposition 3.1 that we may assume without loss of generality that

$$-DG(\rho) \in \mathcal{H}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)\ , \quad \rho \in \mathcal{O},$$

and the existence assertion follows, e.g., from [16, Theorem 8.4.1] and the fact that the little Hölder spaces are stable under continuous interpolation. ◻

Remark 3.9. The above result confirms and quantifies the physical intuition that a solution exists only for small incline angle and ceases to exist for larger angles. Obviously, the determining critical size of the angle depends on Γ_0 and $\mathcal{V}$, i.e., the shape of the initial wetted region Ω_0 and the mass of liquid since these determine the sign of $\partial_{\nu_0} u_0$.

4. Stability analysis for translating circular solutions. Throughout the following we assume that (1.12) is satisfied, that is,

$$F(q) = aq - b, \quad q \in \mathbb{R},$$

for some $a, b > 0$. Recall that if R_0 and v_0 are as in (1.13), then u_0 defined in (1.10) solves (1.6)–(1.9) on the disk $\Omega_* = R_0\mathbb{B}$ (and is positive provided μ is small). We now investigate the asymptotic stability of this translating circular solution for small inclines, i.e., for small $\mu \geq 0$.

4.1. Reformulation. We rewrite problem (1.1)–(1.5) by introducing the translations

$$\tilde{u}(t, x) := u(t, x + tv_0e_1), \quad x \in \tilde{\Omega}(t) := \Omega(t) - tv_0e_1.$$

Substituting this into (1.1)–(1.5) and dropping again the tildes everywhere for ease of notation, it is readily seen that (1.1)–(1.5) is equivalent to

$$-\Delta u = \mu x^1 + \lambda \quad \text{in } \Omega(t), \tag{4.1}$$
$$u = 0 \quad \text{on } \Gamma(t), \tag{4.2}$$
$$\int_{\Omega(t)} u\, dx = \mathcal{V}, \tag{4.3}$$
$$V = F(-\partial_\nu u) - v_0 e_1 \cdot \nu \quad \text{on } \Gamma(t), \tag{4.4}$$
$$\Omega(0) = \Omega^0, \tag{4.5}$$

and (u_0, Ω_*) is a stationary solution to (4.1)–(4.5). As in the previous section, we can consider this problem as a single equation for the geometry. Let $\nu_0(y) = y/R_0$ denote the normal at $y \in \Gamma_0 := R_0\mathbb{S}^1$. Note that in this case, since ν_0 is smooth, we can take $\nu^\delta = \nu_0$ in section 2, and the flow in (2.3) is simply given by $\varphi^\delta(y, r) = (1 + r/R_0)y$. Thus, the evolution of $\Gamma(t)$ is described by the evolution of the function $\rho(t, \cdot) : \Gamma_0 \to \mathbb{R}$ through

$$\Gamma(t) = \big\{ y + \rho(t, y)\nu_0(y) \,\big|\, y \in \Gamma_0 \big\} =: \Gamma_{\rho(t)},$$

and ρ is governed by

$$\begin{cases} \dot\rho = \big(\nu_0 \big| \nu_\rho\big)^{-1}\big(F\big(-\partial_{\nu_\rho} u_\rho\big) - v_0 e_1 \cdot \nu_\rho\big) =: H(\rho), & t > 0, \quad y \in \Gamma_0, \\ \rho(0, \cdot) = \rho_0, \end{cases} \tag{4.6}$$

provided that $\partial\Omega^0 = \Gamma_{\rho_0}$, which is possible for Ω^0 sufficiently close to Ω_* (i.e., for ρ_0 small enough). Here, $\nu_\rho = \nu_{\Gamma_\rho}$, and u_ρ with corresponding λ_ρ denotes the solution of (1.1)–(1.3) from Proposition 2.1 in Ω_ρ, the domain enclosed by Γ_ρ. Clearly, $\Omega_0 = \Omega_* = R_0\mathbb{B}$. Note that

$$H \in C^2(\mathcal{O}, buc^{1+\alpha}(\Gamma_0)), \qquad H(0) = 0, \tag{4.7}$$

according to Proposition 3.1 with $\mathcal{O}$ being an open zero-neighborhood in $buc^{2+\alpha}(\Gamma_0)$, and $\rho = 0$ gives rise to the stationary solution (u_0, Ω_0). We shall prove that $\rho = 0$ is locally asymptotically stable for (4.6) by using the principle of linearized stability.

4.2. Linearization. We now express the Fréchet derivative $DH(0)$ in terms of Fourier expansions. For this we use polar coordinates (r, θ) and Cartesian coordinates x interchangeably and observe that if $h \in buc^{2+\alpha}(\Gamma_0)$, then the form of F implies that

$$DH(0)h = \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(\epsilon h) = -a \frac{d}{d\epsilon}\Big|_{\epsilon=0} \big(\partial_{\nu_{\epsilon h}} u_{\epsilon h}\big) - v_0 \frac{d}{d\epsilon}\Big|_{\epsilon=0} \big(e_1 \cdot \nu_{\epsilon h}\big).$$

Therefore, Lemmas 3.2 and 3.6 entail that

$$\begin{aligned} DH(0)h = {} & aDtN_{\Gamma_0}\big([\partial_{\nu_0} u_0]h\big) - a\big(\nu_0^\mathsf{T} D^2 u_0 \nu_0\big)h - a\partial_{\nu_0} u_0^{(1)} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \lambda_{\epsilon h} \\ & + v_0\big[h\big(d_y\nu_0[\tau_0]\big|\nu_0\big) + h'\big]\tau_0 \cdot e_1 \end{aligned} \tag{4.8}$$

with τ_0 denoting the unit tangent vector to Γ_0. We compute the different terms separately. Let $h \in buc^{2+\alpha}(\Gamma_0)$ be given with representation

$$h(\theta) = \sum_{n\in\mathbb{Z}} \hat h_n e^{in\theta}, \quad \theta \in [0, 2\pi].$$

Then

$$w(r,\theta) = \sum_{n\in\mathbb{Z}} \hat{h}_n \left(\frac{r}{R_0}\right)^{|n|} e^{in\theta}$$

solves

$$-\Delta w = 0 \quad \text{in} \quad \Omega_0, \qquad w = h \quad \text{on} \quad \Gamma_0,$$

and thus

$$DtN_{\Gamma_0}(h) = \partial_r w(R_0,\theta) = \frac{1}{R_0}\sum_{n\in\mathbb{Z}} \hat{h}_n |n| e^{in\theta}.$$

Next, since (1.10) implies

$$\partial_{\nu_0} u_0(R_0,\theta) = \partial_r u_0(R_0,\theta) = -\frac{\mu R_0^2}{8}\left(e^{i\theta}+e^{-i\theta}\right) - \frac{4\mathcal{V}}{\pi R_0^3},$$

it follows that

$$\partial_{\nu_0} u_0 h = -\sum_{n\in\mathbb{Z}} \left[\frac{\mu R_0^2}{8}\left(\hat{h}_{n-1}+\hat{h}_{n+1}\right) + \frac{4\mathcal{V}}{\pi R_0^3}\hat{h}_n\right] e^{in\theta}.$$

In summary, we derive that

$$DtN_{\Gamma_0}\big(\partial_{\nu_0} u_0 h\big) = -\sum_{n\in\mathbb{Z}} \left[\frac{\mu R_0}{8}\left(\hat{h}_{n-1}+\hat{h}_{n+1}\right) + \frac{4\mathcal{V}}{\pi R_0^4}\hat{h}_n\right] |n| e^{in\theta}. \tag{4.9}$$

Similarly, we have from (1.10)

$$\big(\nu_0^{\mathsf{T}} D^2 u_0 \nu_0\big) h = \left(\partial_r^2 u_0\right) h = -\left(\frac{3\mu R_0}{8}\left(e^{i\theta}+e^{-i\theta}\right) + \frac{4\mathcal{V}}{\pi R_0^4}\right) h,$$

hence

$$\big(\nu_0^{\mathsf{T}} D^2 u_0 \nu_0\big) h = -\sum_{n\in\mathbb{Z}} \left[\frac{3\mu R_0}{8}\left(\hat{h}_{n-1}+\hat{h}_{n+1}\right) + \frac{4\mathcal{V}}{\pi R_0^4}\hat{h}_n\right] e^{in\theta}. \tag{4.10}$$

Before computing the third term on the right-hand side of (4.8), we focus on the last term. Clearly, $\big(d_y\nu_0[\tau_0]\big|\nu_0\big) = 0$ so that

$$v_0\big[h\big(d_y\nu_0[\tau_0]\big|\nu_0\big) + h'\big]\tau_0\cdot e_1 = -v_0 \sin\theta h',$$

where the derivative with respect to arc length is $h' = \frac{1}{R_0}\frac{d}{d\theta}h$. Consequently, since $v_0 = \mu a R_0^2/4$ by (1.13),

$$v_0\big[h\big(d_y\nu_0[\tau_0]\big|\nu_0\big) + h'\big]\tau_0\cdot e_1 = -\frac{\mu a R_0}{8}\sum_{n\in\mathbb{Z}} \Big[(n-1)\hat{h}_{n-1} - (n+1)\hat{h}_{n+1}\Big] e^{in\theta}. \tag{4.11}$$

As for the third term on the right-hand side of (4.8) recall formula (3.7) for $\lambda_{\epsilon h}$. Then

$$\begin{aligned}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\lambda_{\epsilon h} = &-\frac{\mathcal{V} - \mu\int_{\Omega_0} u_0^{(x^1)}\,dx}{\left(\int_{\Omega_0} u_0^{(1)}\,dx\right)^2}\ \frac{d}{d\epsilon}\Big|_{\epsilon=0}\left(\int_{\Omega_{\epsilon h}} u_{\epsilon h}^{(1)}\,dx\right)\\ &-\frac{\mu}{\int_{\Omega_0} u_0^{(1)}\,dx}\ \frac{d}{d\epsilon}\Big|_{\epsilon=0}\left(\int_{\Omega_{\epsilon h}} u_{\epsilon h}^{(x^1)}\,dx\right),\end{aligned}$$

and thus, on using

$$u_0^{(x^1)}(r,\theta) = \frac{r}{8}(R_0^2 - r^2)\cos\theta, \qquad u_0^{(1)}(r,\theta) = \frac{1}{4}(R_0^2 - r^2), \qquad \lambda_0 = \frac{8\mathcal{V}}{\pi R_0^4},$$

according to (1.10), we deduce

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \lambda_{\epsilon h} = -\frac{64\mathcal{V}}{\pi^2 R_0^8}\ \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\int_{\Omega_{\epsilon h}} u_{\epsilon h}^{(1)}\,dx\right) - \frac{8\mu}{\pi R_0^4}\ \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\int_{\Omega_{\epsilon h}} u_{\epsilon h}^{(x^1)}\,dx\right).$$

We then invoke the transport theorem to get

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\int_{\Omega_{\epsilon h}} u_{\epsilon h}^{(f)}\,dx\right) = \int_{\Omega_0}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\epsilon h}^{(f)}\,dx + \int_{\Gamma_0} u_0^{(f)} h\,d\sigma_{\Gamma_0} = \int_{\Omega_0}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\epsilon h}^{(f)}\,dx$$

since $u_0^{(f)}$ vanishes on Γ_0. Consequently,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \lambda_{\epsilon h} = -\frac{64\mathcal{V}}{\pi^2 R_0^8}\int_{\Omega_0}\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\epsilon h}^{(1)}\right)dx - \frac{8\mu}{\pi R_0^4}\int_{\Omega_0}\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\epsilon h}^{(x^1)}\right)dx. \tag{4.12}$$

It remains to compute the derivatives for which we proceed as in [11]. Given a smooth function f on $\bar{\Omega}_{\epsilon h}$, let

$$u_{\epsilon h}^{(f)}(x) = \int_{\Omega_{\epsilon h}} K_{\epsilon h}(x,\bar{x}) f(\bar{x})\,d\bar{x}, \quad x \in \bar{\Omega}_{\epsilon h},$$

be a representation of the solution to the Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad \Omega_{\epsilon h}, \qquad u = 0 \quad \text{on} \quad \Gamma_{\epsilon h},$$

with Green's function $K_{\epsilon h}$ on $\Omega_{\epsilon h}$. In particular, for the circle $\Omega_0 = R_0\mathbb{B}$ we have

$$K_0(r,\theta,\bar{r},\bar{\theta}) = \frac{1}{4\pi}\log\left(\frac{R_0^2\left(r^2+\bar{r}^2-2r\bar{r}\cos(\theta-\bar{\theta})\right)}{r^2\bar{r}^2+R_0^4-2R_0^2 r\bar{r}\cos(\theta-\bar{\theta})}\right).$$

Then, noticing that $K_{\epsilon h}$ vanishes on $\Omega_{\epsilon h}\times\Gamma_{\epsilon h}$, we obtain

$$\begin{aligned}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} u_{\epsilon h}^{(f)}(x) &= \int_{\Gamma_0} K_0(x,\bar{x}) f(\bar{x}) h(\bar{x})\,d\sigma_{\Gamma_0}(\bar{x}) + \int_{\Omega_0}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} K_{\epsilon h}(x,\bar{x}) f(\bar{x})\,d\bar{x}\\ &= \int_{\Omega_0}\frac{d}{d\epsilon}\bigg|_{\epsilon=0} K_{\epsilon h}(x,\bar{x}) f(\bar{x})\,d\bar{x}.\end{aligned}$$

Next observe [12, 18] that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} K_{\epsilon h}(x,\bar{x}) = \frac{1}{R_0}\int_0^{2\pi}\partial_r K_0(x,R_0,\phi)\partial_r K_0(\bar{x},R_0,\phi)h(\phi)\,d\phi$$

and that

$$(r,\theta)\mapsto\int_0^{2\pi}\partial_r K_0(r,\theta,R_0,\phi)g(\phi)\,d\phi$$

is the unique harmonic function in Ω_0 with boundary value g on Γ_0. Therefore,

$$\int_0^{2\pi}\partial_r K_0(\bar{r},\bar{\theta},R_0,\phi)\cos\bar{\theta}\,d\bar{\theta} = \frac{\bar{r}}{R_0}\cos\phi$$

since $K_0(\bar r,\bar\theta,\bar r,\phi)$ is symmetric with respect to the angular (and radial) variables. Using this, we obtain

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} u_{\epsilon h}^{(x^1)}(r,\theta) &= \int_{\Omega_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} K_{\epsilon h}(x,\bar x)\bar x^1\, d\bar x \\
&= \frac{1}{R_0}\int_0^{2\pi}\int_0^{R_0} \partial_r K_0(r,\theta,R_0,\phi)\int_0^{2\pi} \partial_r K_0(\bar r,\bar\theta,R_0,\phi)\bar r\cos\bar\theta\, d\bar\theta\, h(\phi)\bar r\, d\bar r\, d\phi \\
&= \frac{1}{R_0^2}\int_0^{2\pi}\int_0^{R_0} \partial_r K_0(r,\theta,R_0,\phi)\bar r^3 h(\phi)\cos\phi\, d\bar r\, d\phi \\
&= \frac{R_0^2}{4}\int_0^{2\pi} \partial_r K_0(r,\theta,R_0,\phi)h(\phi)\cos\phi\, d\phi\,.
\end{aligned}$$

Integrating this yields similarly

$$\begin{aligned}
\int_{\Omega_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_{\epsilon h}^{(x^1)}\, dx &= \frac{R_0^2}{4}\int_0^{2\pi}\int_0^{R_0}\int_0^{2\pi} \partial_r K_0(r,\theta,R_0,\phi)\, d\theta\, r\, dr\, h(\phi)\cos\phi\, d\phi \\
&= \frac{R_0^2}{4}\int_0^{2\pi}\int_0^{R_0} r\, dr\, h(\phi)\cos\phi\, d\phi \;=\frac{R_0^4}{8}\int_0^{2\pi} h(\phi)\cos\phi\, d\phi,
\end{aligned}$$

and finally,

$$\int_{\Omega_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_{\epsilon h}^{(x^1)}\, dx = \frac{\pi R_0^4}{8}\left(\hat h_{-1}+\hat h_1\right). \tag{4.13}$$

In exactly the same way one computes

$$\int_{\Omega_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_{\epsilon h}^{(1)}\, dx = \frac{\pi R_0^3}{2}\hat h_0. \tag{4.14}$$

It now follows from (4.12)–(4.14) that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \lambda_{\epsilon h} = -\frac{32\mathcal{V}}{\pi R_0^5}\,\hat h_0 - \mu\left(\hat h_{-1}+\hat h_1\right). \tag{4.15}$$

Consequently, gathering (4.8)–(4.11) and (4.15) and using $\partial_{\nu_0} u_0^{(1)} = -R_0/2$, we obtain the Fourier expansion of $DH(0)$ in the form

$$\begin{aligned}
DH(0)h = &-\frac{12\mathcal{V}a}{\pi R_0^4}\,\hat h_0 - \frac{4\mathcal{V}a}{\pi R_0^4}\sum_{n\in\mathbb{Z}^*}\big(|n|-1\big)\,\hat h_n\, e^{in\theta} \\
&- \mu\frac{aR_0}{8}\sum_{n\in\mathbb{Z}^*}\Big[\big(|n|+n-4\big)\,\hat h_{n-1} + \big(|n|-n-4\big)\,\hat h_{n+1}\Big]\, e^{in\theta}
\end{aligned} \tag{4.16}$$

for $h\in buc^{2+\alpha}(\Gamma_0)$ with $h(\theta)=\sum_{n\in\mathbb{Z}}\hat h_n e^{in\theta}$, where $\mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$. Note that the matrix is tridiagonal if $\mu>0$. Since $DH(0)$ has a compact resolvent, its spectrum is discrete and contains only eigenvalues. More information is found in the following lemma.

LEMMA 4.1. *The kernel of $DH(0)$ is two-dimensional and spanned by $\{e^{-i\theta}, e^{i\theta}\}$. Moreover, there is $\omega>0$ independent of μ such that if $\mu\ge 0$ is small enough, then $\sigma(DH(0))\subset[\mathrm{Re}z\le-\omega]\cup\{0\}$.*

Proof. It follows from (4.16) using an induction argument that $h \in buc^{2+\alpha}(\Gamma_0)$ with $h = \sum_{n\in\mathbb{Z}} \hat{h}_n e^{in\theta}$ belongs to the kernel of $DH(0)$ if and only if $\hat{h}_n = 0$ for $n \in \mathbb{Z} \setminus \{\pm 1\}$. Hence the kernel of $DH(0)$ is spanned by $\{e^{-i\theta}, e^{i\theta}\}$. Next,

$$DH(0) = A + \mu B$$

with $-A \in \mathcal{H}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$ (see Theorem 3.7 with $\mu = 0$ therein) given by

$$Ah := -\frac{12\mathcal{V}a}{\pi R_0^4}\,\hat{h}_0 - \frac{4\mathcal{V}a}{\pi R_0^4}\sum_{n\in\mathbb{Z}^*}\big(|n|-1\big)\,\hat{h}_n\,e^{in\theta} \tag{4.17}$$

and $B \in \mathcal{L}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$ given by

$$Bh := -\frac{aR_0}{8}\sum_{n\in\mathbb{Z}^*}\Big[\big(|n|+n-4\big)\,\hat{h}_{n-1} + \big(|n|-n-4\big)\,\hat{h}_{n+1}\Big]\;e^{in\theta}. \tag{4.18}$$

Set $\omega := \frac{\mathcal{V}a}{\pi R_0^4}$. Since $\sigma(A) = -4\omega\mathbb{N}$, [1, I.Corollary 1.4.3] implies that there are $\mu_0 > 0$ and $\vartheta \in (0, \pi/2)$ such that $\sigma(A + \mu B) \subset \omega + \Sigma_\vartheta$ for $\mu \in [0, \mu_0)$, where

$$\Sigma_\vartheta := [\arg(z) \in (\pi - \vartheta, \pi + \vartheta)]\,.$$

Since zero is the only eigenvalue of A in $\Sigma_\vartheta \cap [\mathrm{Re}\, z > -\omega]$ and since $A + \mu B \to A$ in the generalized sense of [14, IV.Theorem 2.24] as $\mu \to 0$, it follows from [14, IV.Theorem 3.16] that zero is the only eigenvalue of $DH(0) = A + \mu B$ in $\Sigma_\vartheta \cap [\mathrm{Re}\, z > -\omega]$ if $\mu \geq 0$ is sufficiently small. This proves the assertion. ∎

4.3. Stability analysis. To analyze now the stability of the zero solution to (4.6) we need to split off the zero eigenvalue of the linearization $DH(0)$. For this it is useful to use a slightly different description of the curves Γ_ρ as provided by the next lemma.

LEMMA 4.2. *For each $\rho \in \mathcal{O}$ there is a unique $(z, \bar{\rho}) \in \mathbb{R}^2 \times \big(\ker(DH(0))\big)^\perp$ such that $\Gamma_\rho = z + \Gamma_{\bar{\rho}}$. Moreover, $H(\rho) = H(\bar{\rho})$.*

Proof. Since, due to Lemma 4.1, the kernel of $DH(0)$ is spanned by $\{e^{-i\theta}, e^{i\theta}\}$, the existence of a unique $(z, \bar{\rho}) \in \mathbb{R}^2 \times (\ker(DH(0)))^\perp$ with $\Gamma_\rho = z + \Gamma_{\bar{\rho}}$ is shown in [11, Lemma 5.2]. That $H(\rho) = H(\bar{\rho})$ follows as in [11, Lemma 5.1] from the translation invariance of the problem. Indeed, if $\Gamma = \partial\Omega$ and u_Γ solves

$$\begin{aligned} -\Delta u &= \mu x^1 + \lambda \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ \int_\Omega u\,dx &= \mathcal{V}, \end{aligned}$$

then $u_{\Gamma_{\bar{\rho}}} = u_{\Gamma_\rho}(\cdot + z)$ on $\Omega_{\bar{\rho}} = -z + \Omega_\rho$ and $\nu_{\Gamma_{\bar{\rho}}} = \nu_{\Gamma_\rho}(\cdot + z)$ on $\Gamma_{\bar{\rho}} = -z + \Gamma_\rho$. The definition of H implies the assertion. ∎

We next derive the evolution in the new coordinates $(z, \bar{\rho}) = (z(\rho), \bar{\rho}(\rho))$. Let $\nu_0 = (\cos, \sin)$ and $\tau_0 = (-\sin, \cos)$ denote the unit normal and unit tangent vector to Γ_0, respectively, and let

$$p = (R_0 + \rho(t, \theta))\nu_0(\theta) = z + (R_0 + \bar{\rho}(t, \phi))\nu_0(\phi)$$

be an arbitrary point on $\Gamma_{\rho(t,\cdot)} = z + \Gamma_{\bar\rho(t,\cdot)}$. We often suppress the time variable t in the following. Differentiating with respect to ϕ implies that

$$\nu_{\Gamma_\rho}(p) = \frac{-\partial_\phi\bar\rho(\phi)\tau_0(\phi) + (R_0+\bar\rho(\phi))\nu_0(\phi)}{\big((\partial_\phi\bar\rho(\phi))^2 + (R_0+\bar\rho(\phi))^2\big)^{1/2}}. \tag{4.19}$$

Now, since $V_{\Gamma_\rho}(p) = \big(\dot z + \dot{\bar\rho}(\phi)\nu_0(\phi)\big)\cdot\nu_\rho(p)$ with dots indicating time derivatives, it follows from the definition of H and $H(\bar\rho) = H(\rho)$ that

$$H(\bar\rho)(p) = \frac{1}{\nu_0(\phi)\cdot\nu_\rho(p)}\big(\dot z + \dot{\bar\rho}(\phi)\nu_0(\phi)\big)\cdot\nu_\rho(p).$$

Therefore, from (4.19) we derive that

$$H(\bar\rho) = \left(\frac{\partial_\phi\bar\rho(\phi)}{R_0+\bar\rho(\phi)}\sin\phi + \cos\phi\right)\dot z_1 + \left(-\frac{\partial_\phi\bar\rho(\phi)}{R_0+\bar\rho(\phi)}\cos\phi + \sin\phi\right)\dot z_2 + \dot{\bar\rho}. \tag{4.20}$$

Let $\Pi^{(\pm1)} \in \mathcal{L}(buc^{1+\alpha}(\Gamma_0))$ denote the projections onto the subspaces spanned by $e^{\pm i\theta}$, that is,

$$\Pi^{(\pm1)}f := \frac{1}{2\pi}\int_0^{2\pi} f(\vartheta)e^{\mp i\vartheta}\,d\vartheta,$$

and let $\Pi^\perp \in \mathcal{L}(buc^{1+\alpha}(\Gamma_0))$ denote the projection onto $\big(\ker(DH(0))\big)^\perp$. We then apply these projections to (4.20) to derive the evolution for $(z,\bar\rho)$. For a more compact notation we introduce

$$\begin{aligned} M(\bar\rho) :=& \begin{bmatrix} \frac12 & \frac{1}{2i} \\ \frac12 & -\frac{1}{2i}\end{bmatrix} \\ &+ \begin{bmatrix} \frac{1}{4\pi i}\int_0^{2\pi}\frac{\partial_\phi\bar\rho(\phi)}{R_0+\bar\rho(\phi)}\left(1-e^{-2\phi i}\right)d\phi & -\frac{1}{4\pi}\int_0^{2\pi}\frac{\partial_\phi\bar\rho(\phi)}{R_0+\bar\rho(\phi)}\left(e^{-2\phi i}+1\right)d\phi \\ \frac{1}{4\pi i}\int_0^{2\pi}\frac{\partial_\phi\bar\rho(\phi)}{R_0+\bar\rho(\phi)}\left(e^{2\phi i}-1\right)d\phi & -\frac{1}{4\pi}\int_0^{2\pi}\frac{\partial_\phi\bar\rho(\phi)}{R_0+\bar\rho(\phi)}\left(e^{2\phi i}+1\right)d\phi \end{bmatrix}\end{aligned}$$

and observe that

$$M(\bar\rho) = \begin{bmatrix} \frac12 & \frac{1}{2i} \\ \frac12 & -\frac{1}{2i}\end{bmatrix} + O\big(\|\bar\rho\|_{buc^{2+\alpha}(\Gamma_0)}\big)$$

is invertible if $\bar\rho\in\mathcal{O}$ is small. Further set

$$\Pi(H(\bar\rho)) := \begin{pmatrix}\Pi^{(+1)}(H(\bar\rho)) \\ \Pi^{(-1)}(H(\bar\rho))\end{pmatrix}$$

and

$$f(\bar\rho) := \Pi^\perp\left(\frac{\partial_\phi\bar\rho}{R_0+\bar\rho}\tau_0\cdot M(\bar\rho)^{-1}\Pi(H(\bar\rho))\right).$$

Then we obtain from (4.20) the following proposition.

PROPOSITION 4.3. *The evolution for ρ governed by* (4.6) *is equivalent to the system*

$$\dot z = M(\bar\rho)^{-1}\Pi(H(\bar\rho)), \tag{4.21}$$

$$\dot{\bar\rho} = \Pi^\perp H(\bar\rho) + f(\bar\rho) \tag{4.22}$$

for $(z,\bar\rho)$.

Next, we introduce

$$buc^{k+\alpha}_{\perp}(\Gamma_0) := \Pi^{\perp} buc^{k+\alpha}(\Gamma_0), \quad k = 1, 2.$$

Then $\mathcal{O}_{\perp} := \mathcal{O} \cap buc^{2+\alpha}_{\perp}(\Gamma_0)$ is an open zero-neighborhood in $buc^{2+\alpha}_{\perp}(\Gamma_0)$ and (4.7) entails that

$$f \in C^2\big(\mathcal{O}_{\perp}, buc^{1+\alpha}_{\perp}(\Gamma_0)\big) \quad \text{with} \quad f(0) = 0,\ Df(0) = 0. \tag{4.23}$$

Moreover, for the linearization of $\Pi^{\perp} H \in C^2\big(\mathcal{O}_{\perp}, buc^{1+\alpha}_{\perp}(\Gamma_0)\big)$ at zero we have the following proposition.

PROPOSITION 4.4. *Let $\mu \geq 0$ be sufficiently small. Then*

$$-D\big(\Pi^{\perp} H\big)(0) \in \mathcal{H}\big(buc^{2+\alpha}_{\perp}(\Gamma_0), buc^{1+\alpha}_{\perp}(\Gamma_0)\big),$$

and there is $\omega_0 > 0$ such that

$$\sigma\big(D\big(\Pi^{\perp} H\big)(0)\big) \subset [\mathrm{Re} z \leq -2\omega_0].$$

Proof. Observe that, owing to (4.17), (4.18),

$$D\big(\Pi^{\perp} H\big)(0) = \Pi^{\perp} DH(0) = A_{\perp} + \mu B_{\perp},$$

where

$$A_{\perp} h := -\frac{12\mathcal{V}a}{\pi R_0^4}\, \hat{h}_0 - \frac{4\mathcal{V}a}{\pi R_0^4} \sum_{\substack{n\in\mathbb{Z}^*\\ n\neq\pm1}} \big(|n| - 1\big)\, \hat{h}_n\, e^{in\theta}$$

and

$$B_{\perp} h := -\frac{aR_0}{8} \sum_{\substack{n\in\mathbb{Z}^*\\ n\neq\pm1}} \Big[\big(|n| + n - 4\big)\, \hat{h}_{n-1} + \big(|n| - n - 4\big)\, \hat{h}_{n+1}\Big]\, e^{in\theta}$$

for $h \in buc^{2+\alpha}_{\perp}(\Gamma_0)$ with $h(\theta) = \sum_{n\in\mathbb{Z},\, n\neq\pm1} \hat{h}_n e^{in\theta}$. Since

$$-A \in \mathcal{H}\big(buc^{2+\alpha}(\Gamma_0), buc^{1+\alpha}(\Gamma_0)\big)$$

by Theorem 3.7 (with $\mu = 0$), one readily deduces from the obvious fact

$$A_{\perp} = A\big|_{buc^{2+\alpha}_{\perp}(\Gamma_0)}$$

that $-A_{\perp} \in \mathcal{H}\big(buc^{2+\alpha}_{\perp}(\Gamma_0), buc^{1+\alpha}_{\perp}(\Gamma_0)\big)$. Observing then that

$$\sigma(A_{\perp}) = \sigma(A) \setminus \{0\} = -4\omega\mathbb{N} \setminus \{0\}, \quad \omega := \frac{\mathcal{V}a}{\pi R_0^4},$$

the assertion follows from the perturbation result [1, I.Proposition 1.4.2]. ∎

Due to Proposition 4.4 and (4.23) we are in a position to apply the principle of linearized stability from [16, Theorem 9.1.2] to (4.22) for $\bar{\rho}$. Thus, there are $r > 0$ and $M > 0$ such that for any initial value $\bar{\rho}_0 \in buc^{2+\alpha}_{\perp}(\Gamma_0)$ with $\|\bar{\rho}_0\|_{buc^{2+\alpha}_{\perp}} < r$, the solution $\bar{\rho}$ to (4.22) with $\bar{\rho}(0) = \bar{\rho}_0$ exists globally in time and

$$\|\bar{\rho}(t)\|_{buc^{2+\alpha}_{\perp}} + \|\dot{\bar{\rho}}(t)\|_{buc^{1+\alpha}_{\perp}} \leq Me^{-\omega_0 t}, \quad t \geq 0. \tag{4.24}$$

Plugging this into (4.21) and observing that the right-hand side of (4.21) is of order $O\big(\|\bar\rho\|_{buc_\perp^{2+\alpha}}\big)$ owing to (4.7), it readily follows that there is $z_\infty \in \mathbb{R}^2$ such that

$$|z(t) - z_\infty| \le ce^{-\omega_0 t}, \quad t \ge 0. \tag{4.25}$$

Therefore, using again the original coordinates instead of $(z, \bar\rho)$ and noticing that $\|\bar\rho(\rho)\|_{buc_\perp^{2+\alpha}}$ is small if and only if $\|\rho\|_{buc^{2+\alpha}}$ is, we arrive at the following theorem.

THEOREM 4.5. *Assume that* (1.12) *holds. Then, for small incline* $\mu > 0$*, the stationary solution* (u_0, Ω_*) *to* (4.1)–(4.5) *from* (1.10) *is asymptotically exponentially stable. More precisely, if* $\mu > 0$ *is small and given an initial geometry*

$$\Gamma_{\rho_0} = \{(R_0 + \rho_0(\theta))e^{i\theta}\}$$

with $\|\rho_0\|_{buc^{2+\alpha}(\Gamma_0)}$ *sufficiently small, there exist*

$$\rho \in C^1\big(\mathbb{R}^+, buc^{1+\alpha}(\Gamma_0)\big) \cap C\big(\mathbb{R}^+, buc^{2+\alpha}(\Gamma_0)\big)$$

and u_ρ *with*

$$u_{\rho(t)} \in buc^{1+\alpha}\big(\Omega_{\rho(t)}\big), \quad \partial\Omega_{\rho(t)} = \Gamma_{\rho(t)}$$

for $t \ge 0$ *such that* (ρ, u_ρ) *satisfies* (4.6)*. Moreover,*

$$\Gamma_{\rho(t)} = z(t) + \Gamma_{\bar\rho(t)}, \quad t \ge 0,$$

with

$$\bar\rho \in C^1\big(\mathbb{R}^+, buc^{1+\alpha}(\Gamma_0)\big) \cap C\big(\mathbb{R}^+, buc^{2+\alpha}(\Gamma_0)\big), \qquad z \in C^1\big(\mathbb{R}^+, \mathbb{R}^2\big)$$

satisfying (4.24) *and* (4.25)*.*

The above theorem shows that, for small inclines and when starting out with a droplet geometry sufficiently close to the disk of radius R_0, the droplet asymptotically becomes circular of radius R_0 sliding down the plane with constant speed v_0.

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