# BOUNDARY DYNAMICS OF A TWO-DIMENSIONAL DIFFUSIVE FREE BOUNDARY PROBLEM

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ABSTRACT. Numerous models of industrial processes such as diffusion in glassy polymers or solidification phenomena, lead to general one-phase free boundary value problems with phase onset. In this paper we develop a framework viable to prove global existence and stability of planar solutions to one such multi-dimensional model whose application is in controlled-release pharmaceuticals. We utilize a boundary integral reformulation to allow for the use of maximal regularity. To this effect, we view the operators as pseudo-differential and exploit knowledge of the relevant symbols. Within this framework, we give a local existence and continuous dependence result necessary to prove planar solutions are locally exponentially stable with respect to two-dimensional perturbations.

1. **Introduction.** Over the past several decades, modeling solute diffusion in glassy polymers has been a topic of interest with direct application to industry. For example, in controlled-release pharmaceuticals, it is important to understand how a drug will diffuse through its storage device, a polymer. Due to the large class of polymers with different properties, no single model for polymer diffusion exists. Rather, different models are used for groups of polymers sharing the same characteristics. Certain polymers exhibit a behavior called Case II diffusion; it is defined by the onset of a sharp discontinuity within the polymer which separates a glassy region with negligible solute concentration from a swollen rubbery region with high solute concentration. In addition, Case II diffusion is said to occur when the front initially travels with near constant speed. Alfrey, Gurnee, and Lloyd were among the first researchers to consider the mathematical modeling of swelling polymers under diffusion different then the classical $\sqrt{t}$ Fickian diffusion (see [2]). Researchers have proposed several models for Case II diffusion; here we focus on the approach proposed by Astarita and Sarti in [4]. They captured the features of non-Fickian

diffusion by incorporating a phenomenological law into a one-dimensional Stefan-like moving boundary problem for the solute concentration and penetration depth.

In their paper [5], Cohen and Erneux completed and extended the analysis of Astarita and Sarti by describing the complete history of the penetrant front. They proved there is a transition from Case II diffusion behavior, where the front moves proportional to time, to Fickian diffusion, where the front moves proportional to the square-root of time. Furthermore, their asymptotic analysis of the one-dimensional model showed the dependence on the various parameters of the problem which allows them to give optimal strategies in pharmaceutical application.

As we all know, polymers are three-dimensional, so a model in higher dimensions would be of great benefit. In the case that the polymer is very thin, such as in pharmaceuticals, a two-dimensional model is of interest. In [6], Guidotti and Pelesko proposed a two-dimensional generalization of the Astarita-Sarti model and developed an asymptotic theory. They proved that one-dimensional planar fronts are asymptotically stable under infinitesimal perturbations, yet transient instability may be present. In [8], Guidotti extended the model to account for curvature effects on the speed of the front and concentration of the solute. He proved existence and uniqueness to the solution of the quasi-stationary approximation to the two-dimensional model with and without curvature. In [10], Guidotti extended his previous result by proving well-posedness for the full evolutionary problem.

In light of Guidotti's results it is natural to ask the question of global existence of the concentration profile and sharp interface. In the one-dimensional Case II diffusion model we know solutions persist for all time, and we take advantage of this fact in the two-dimensional model given by Guidotti in [6] by making the following observation. When the two-dimensional model is given constant boundary datum, the model reduces to the one-dimensional one, and hence the system has a global solution. The planar or flat solutions become the basis of our analysis. We consider the effect two-dimensional perturbations have on flat solutions. In an appropriate setting where the principle of linearized stability holds, we seek to prove flat solutions are asymptotically stable.

In order to apply the principle of linearized stability we must reformulate the model to consider only the dynamics of the model on the free boundary. There are two reasons for this: first, the governing equation for the free boundary is coupled to the concentration profile; second, a boundary condition prevents stability on the whole domain. To obtain an equivalent system for the boundary effects, we employ a boundary integral formulation. The approach has several advantages for performing nonlinear analysis: first, essentially we make the problem one-dimensional. Second, we can write the concentration on the free-boundary as an operator depending solely on the free-boundary. Most importantly, the boundary integral formulation allows us to view the problem as a dynamical system where we can exploit maximal regularity to obtain local and global existence results for the free-boundary. The key element of many theorems from analytic semigroup theory using maximal regularity arguments is given by the properties of the generator. To analyze the behavior of the generator we view the boundary integral formulation through the lens of pseudo-differential operators. In this setting we can explicitly compute the resolvent and spectrum of the generator and use existence theorems.

All of the analysis in this paper is motivated by developing a framework viable for proving asymptotic stability of planar solutions. In the process of development, we prove local existence and continuous dependence of a class of solutions different

than that proved in [8]. Furthermore, we need the local existence result as stated in this paper to continue on to prove the existence of global solutions to the multi-dimensional Case II diffusion model. We organize the paper as follows: in Section 2 we formulate the two-dimensional model and briefly discuss the planar or flat solutions. In Section 3 we derive a system to capture the boundary behavior of the Case II diffusion model. In Section 4 we state the main result of the paper, Theorem 4.1. Sections 5-6 are devoted to proving Theorem 4.1. In these sections, we perform an in-depth analysis of the operators' symbols in order to show that the formulation is well-defined and to derive properties used in the existence and stability theorems as given in [12].

2. **Formulation of the 2-D case II diffusion model.** We assume that a polymer half-space is exposed to a reservoir of solute comprised of small molecules capable of diffusing into the polymer. In the half-space, a sharp interface separates the polymer in two parts. The first is a swollen rubbery region, $\Omega_t$, where we assume the solute is free to diffuse, and the second is a glassy region with negligible concentration of solute. We skip the non-dimensionalization process and construction of the quasi-stationary approximation and refer to [6] and [8]. The quasi-stationary model for Case II diffusion is given by:

$$-\triangle u = 0, \qquad\qquad\qquad \text{in } \Omega_t, \qquad\qquad (1)$$

$$\gamma_0 u = g > 0, \qquad\qquad\qquad \text{on } \Gamma_0 \qquad\qquad (2)$$

$$-\partial_\nu u = \dot{s}, \qquad\qquad\qquad \text{on } \Gamma_t \qquad\qquad (3)$$

$$\dot{s} = (\sqrt{1 + s_x^2} + \delta\frac{s_{xx}}{1 + s_x^2})\gamma_t u, \quad \text{on } \Gamma_t \qquad\qquad (4)$$

$$s(0) = 0 \qquad\qquad\qquad \text{on } \Gamma_0 \qquad\qquad (5)$$

Equation (1) must hold in an unknown, strip-like domain $\Omega_t$ with fixed boundary $\Gamma_0 = \mathbb{R} \times \{0\}$ and moving boundary

$$\Gamma_t = \{(x, s(t, x)) | x \in \mathbb{R}\}$$

for positive time. The notation $\gamma_i$ represents the restriction operator on $\Gamma_i$ for $i = 0, t$, and $\nu$ is the outward normal vector. We assume that $\Gamma_t$ can be parametrized by an unknown smooth function $s$. In addition, $u$ denotes the concentration of solute and $\delta$ is a nonnegative parameter. A consequence of (1)-(5) is $\dot{s}(0) = g$, and even though this is not a necessary condition we will at times use it for clarity.

When the boundary datum is flat the system yields a flat-solution pair denoted by $(u_f(t, y), s_f(t))$ or just $(u_f, s_f)$. Here, flat refers to the solution pair being independent of the spatial $x$-variable. In this case (1)-(5) reduce to the following

$$-\partial_{yy} u = 0 \qquad\qquad\qquad\qquad (6)$$

$$u(0, t) = g_f \qquad\qquad\qquad\qquad (7)$$

$$\dot{s} = -\partial_y u(s(t), t) \qquad\qquad\qquad\qquad (8)$$

$$\dot{s} = u(s(t), t) \qquad\qquad\qquad\qquad (9)$$

$$s(0) = 0 \qquad\qquad\qquad\qquad (10)$$

Using the Ansatz that $u$ is linear in $y$, it is easy to see that

$$u(y, t) = c_1(t)y + g_f$$

satisfies (6) and (7). Now substituting this representation for $u$ into (8) and using (9) we see

$$\dot{s}(1 + s) = g_f$$
$$s(0) = 0$$

Hence we have a solution pair defined as follows

$$s(t) = -1 + \sqrt{1 + 2g_f t} =: s_f(t)$$
$$u(y, t) = (\frac{-g_f}{\sqrt{1 + 2g_f t}})y + g_f =: u_f(y, t)$$

The pair $(u_f, s_f)$ satisfies (1)-(5), and we should point out that with flat boundary datum we can apply the uniqueness result derived by Guidotti in [8] to justify the simplification made in (6)-(10).

We should remark that due to condition (2), once we allow $x$-dependence in the boundary datum it is impossible for the solution pair of (1)-(5) to converge to the flat solution pair of (6)-(7). This observation led us to the idea of using a boundary integral formulation (BIF) to analyze the behavior of solution pair only on $\Gamma_t$. In this setting, we can investigate stability properties of the solution restricted to $\Gamma_t$.

Let $\Omega$ be a domain in $\mathbb{R}^n$. We denote by $L_2(\Omega)$ the usual Hilbert Space of square-integrable functions on $\Omega$. $H^m(\Omega)$ will denote the class of all functions defined on $\Omega$ whose first $m$ weak derivatives are in $L_2(\Omega)$, and with norm

$$\|f\|^2_{H^m(\Omega)} := \sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|^2_{L_2}$$

Let $X$ be a real or complex Banach space with norm $\|\cdot\|$, and let $J \subset \mathbb{R}$ be an interval, and define the functional space $B(J; X)$ as the collection of all bounded functions $f : J \to X$. We endow $B(J; X)$ with the sup norm

$$\|f\|_{B(J;X)} = \sup_{t \in J} \|f(t)\|$$

We also define the space of bounded continuous, $m$ times continuously differentiable functions as follows

$$C_b(J; X) = B(J; X) \cap C(J; X), \quad \|f\|_{C_b(J;X)} = \|f\|_{B(J;X)}$$
$$C_b^m(J; X) = \{f \in C^m(J; X) : f^{(k)} \in C_b(J; X), k = 0, 1, \ldots m\}$$
$$\|f\|_{C_b^m(J;X)} = \sum_{k=0}^{m} \|f^{(k)}\|_{B(J;X)}$$

The main reason for introducing the previous spaces is to define the spaces of Hölder continuous functions: $C^\alpha(J; X)$, $C^{k+\alpha}(J; X)$ for $k \in \mathbb{N}, \alpha \in (0, 1)$

$$C^\alpha(J; X) = \{f \in C_b(J; X); [f]_{C^\alpha(J;X)} = \sup_{t,s \in J, s < t} \frac{\|f(t) - f(s)\|}{(t - s)^\alpha} < +\infty\}$$
$$\|f\|_{C^\alpha(J;X)} = \|f\|_{B(J;X)} + [f]_{C^\alpha}$$
$$C^{k+\alpha}(J; X) = \{f \in C_b^k(J; X) : f^{(k)} \in C^\alpha(J; X)\}$$
$$\|f\|_{C^{k+\alpha}(J;X)} = \|f\|_{C_b^k(J;X)} + [f^{(k)}]_{C^\alpha(J;X)}$$

3. **Boundary integral formulation.** Unless stated otherwise we assume periodicity conditions for the concentration, $u$, and the sharp interface, $s$. We introduce the 1-periodic Green's function given by Guidotti in [9]

$$G(x, y) = \frac{1}{2\pi} \log |1 + e^{-4\pi y} - 2\cos(2\pi x)e^{-2\pi y}|$$

corresponding to

$$-\triangle G = \delta_{(x,y)}, \quad (x, y) \in [0, 1) \times [0, \infty)$$

in the derivation of the integral equations. Assume $t$ is fixed nonnegative, $u$ satisfies

$$-\triangle u = 0 \quad \text{on } \Omega_t$$

then by Green's identity we have:

$$u(z) = \int_{\Gamma_0 \cup \Gamma_t} \left[ G(z - \tilde{z}) \frac{\partial u}{\partial \nu}(\tilde{z}) - u(\tilde{z}) \frac{\partial G}{\partial \nu}(z - \tilde{z}) \right] dS(\tilde{z}), \quad z \in \Omega_t \qquad (11)$$

Using the parameterizations, $\Gamma_0 = \big\{ (\tilde{x}, 0) | \tilde{x} \in [0, 1] \big\}$ and

$$\Gamma_t = \big\{ (1 - \tilde{x}, s(1 - \tilde{x})) | \tilde{x} \in [0, 1] \big\}$$

combined with independence of direction of arc length line integrals (11) yields

$$\begin{aligned}
u(x, y) = & \int_0^1 \left[ g(\tilde{x}) \frac{\partial G}{\partial y}(x - \tilde{x}, y) - \frac{\partial u}{\partial y}(\tilde{x}, 0) G(x - \tilde{x}, y) \right] d\tilde{x} \\
& - \int_0^1 G(x - \tilde{x}, y - s(\tilde{x})) \dot{s}(\tilde{x}) d\tilde{x} \qquad (12) \\
& - \int_0^1 u(\tilde{x}, s(\tilde{x})) (-s_{\tilde{x}}(\tilde{x}), 1) \cdot \nabla G(x - \tilde{x}, y - s(\tilde{x})) d\tilde{x}
\end{aligned}$$

An important equality needed for the BIF is

$$\frac{\partial G}{\partial y}(x - \tilde{x}, 0^+) = \delta(x - \tilde{x}) - 1$$

(see [9]). Analyzing (12) when $(x, y) = (x, 0)$ and $(x, y) = (x, s(x))$ yields the following BIF to (1)-(5).

**Boundary Integral Formulation.** *Let* $\Psi(x) = u(x, s(x))$, *and*

$$\Phi(x) = -\frac{\partial u}{\partial y}(x, 0) - \dot{s}(x)$$

*then a smooth* $[0, 1]$-*periodic solution of* (1)-(5) *evaluated on* $\Gamma_0$ *and* $\Gamma_t$ *satisfies*

$$\int_0^1 G(\cdot - \tilde{x}, 0)\Phi(\tilde{x})d\tilde{x} = \mathcal{R}_1(g, s, \dot{s}) + \int_0^1 (-s_x(\tilde{x}), 1) \cdot \nabla G(\cdot - \tilde{x}, -s(\tilde{x}))\Psi(\tilde{x})d\tilde{x}$$
$$(13)$$

$$\Psi + \int_0^1 (-s_x(\tilde{x}), 1) \cdot \nabla G(\cdot - \tilde{x}, s(\cdot) - s(\tilde{x}))\Psi(\tilde{x})d\tilde{x}$$
$$= \int_0^1 G(\cdot - \tilde{x}, s(\cdot))\Phi(\tilde{x})d\tilde{x} + \mathcal{R}_2(g, s, \dot{s}) \quad (14)$$

$$\dot{s} = (\sqrt{1 + s_x^2} + \delta \frac{s_{xx}}{1 + s_x^2})\Psi \tag{15}$$

$$s(0) = 0 \tag{16}$$

$$\dot{s}(0) = g \tag{17}$$

*where*

$$\mathcal{R}_1(g, s, \dot{s})(x) = \int_0^1 \left[ g(\tilde{x}) + \left( G\big(x - \tilde{x}, -s(\tilde{x})\big) - G(x - \tilde{x}, 0) \right) \dot{s}(\tilde{x}) \right] d\tilde{x}$$

$$\mathcal{R}_2(g, s, \dot{s})(x) = \int_0^1 \left[ G\big(x - \tilde{x}, s(x)\big) - G\big(x - \tilde{x}, s(x) - s(\tilde{x})\big) \right] \dot{s}(\tilde{x}) d\tilde{x}$$

$$+ \int_0^1 g(\tilde{x}) \frac{\partial G}{\partial y} \big(x - \tilde{x}, s(x)\big) d\tilde{x}$$

We begin our analysis of (13)-(17) by investigating the solvability of (13) assuming $s$ and $\Psi$ are given. To this end we introduce the Fourier transform on the circle and cite a result proved in [9] regarding $G(x, y)$. Denote the Fourier Transform on the circle $[0, 1]$ by

$$(\mathcal{F}_{x \mapsto k} u)(k) = \hat{u}(k) = \int_0^1 e^{-2\pi i k x} u(x) dx, \quad k \in \mathbb{Z}$$

and the Fourier inverse by

$$(\mathcal{F}_{k \mapsto x}^{-1} u)(x) = \sum_{k \in \mathbb{Z}} \hat{u}(k) e^{2\pi i k x}, \quad x \in [0, 1]$$

Let $H_p^m = H_p^m([0, 1])$ be the Sobolev spaces on the unit circle, that is,

$$H_p^m([0, 1]) = \{ u \in L_2([0, 1]) | \sum_{k \in \mathbb{Z}} (1 + |k|)^m |\hat{u}(k)|^2 < \infty \}$$

We denote $H_p^0$ by $L_{2,p}$ and we set

$$\|u\|_{H_p^m}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|)^m |\hat{u}(k)|^2$$

In his paper [9] Guidotti considered the periodic Dirichlet-to-Neumann map as a pseudo-differential operator represented through boundary integrals. He used the same logarithmic kernel, $G(x, y)$, and was confronted with solving an integral equation of the first kind similar to (13). He showed that given $h \in H_p^1$ the integral equation

$$\int_0^1 G(x - \tilde{x}, 0) f(\tilde{x}) d\tilde{x} = h(x)$$

has the unique mean zero solution in $L_{2,p}$ given by

$$f = \mathcal{N}[h] := \mathcal{F}^{-1}[2\pi |k|] \mathcal{F}[h]$$

Assuming the right hand side of (13) is in $H_p^1$ we can use this result to solve for $\Phi$ provided it has mean zero. This fact follows immediately from the necessity that

$$\int_0^1 \left[ -\frac{\partial u}{\partial y}(\tilde{x}, 0) - \dot{s}(\tilde{x})) \right] d\tilde{x} = 0$$

Later we make this precise, but formally we can solve for $\Phi$ in terms of $s$ and $\Psi$ via Fourier Transform.

$$\Phi(x) = \mathcal{N}\left[\mathcal{R}_1(g, s, \dot{s})(\cdot) + \int_0^1 \big(-s_x(\tilde{x}), 1\big) \cdot \nabla G\big(\cdot - \tilde{x}, -s(\tilde{x})\big)\Psi(\tilde{x})d\tilde{x}\right](x) \quad (18)$$

Using (18) in (14), $\Psi$ must solve the integral equation

$$\Psi + \int_0^1 \big(-s_x(\tilde{x}), 1\big)\nabla G\big(\cdot - \tilde{x}, s(\cdot) - s(\tilde{x})\big)\Psi(\tilde{x})d\tilde{x}$$

$$-\int_0^1 G\big(\cdot - \tilde{x}, s(\cdot)\big)\mathcal{N}\left[\int_0^1 \big(-s_{\tilde{x}}(z), 1\big) \cdot \nabla G\big(\tilde{x} - z, -s(z)\big)\Psi(z)dz\right]d\tilde{x} = \mathcal{R}_3(g, s, \dot{s}) \quad (19)$$

where

$$\mathcal{R}_3(g, s, \dot{s}) = \int_0^1 G\big(\cdot - \tilde{x}, s(\cdot)\big)\mathcal{N}\left[\mathcal{R}_1(g, s, \dot{s})(\cdot)\right](\tilde{x})d\tilde{x} + \mathcal{R}_2(g, s, \dot{s}) \quad (20)$$

A natural question is whether or not we can solve (19) for $\Psi$. Since our goal is to prove local existence we focus on the situation when we assume $t \approx 0$ where we can show that the integral operator possesses an inverse.

4. **Analytical results.** Let $s \in H_p^2([0,1]), \dot{s} \in L_{2,p}([0,1]), g \in H_p^2([0,1])$ and consider the boundary integral equation for $\Psi[s](x) := u\big(x, s(x)\big)$ given by

$$(\mathrm{id} + \mathcal{I}_s)[\Psi] = \mathcal{R}_3(g, s, \dot{s})$$

where

$$\mathcal{I}_s[\Psi](x) = \int_0^1 \big(-s_x(\tilde{x}), 1\big)\nabla G\big(x - \tilde{x}, s(x) - s(\tilde{x})\big)\Psi(\tilde{x})d\tilde{x}$$

$$-\int_0^1 G\big(x - \tilde{x}, s(x)\big)\mathcal{N}\left[\int_0^1 \big(-s_{\tilde{x}}(z), 1\big) \cdot \nabla G\big(\tilde{x} - z, -s(z)\big)\Psi(z)dz\right]d\tilde{x}$$

and $\mathcal{R}_3$ is defined in (20). Then the equivalent boundary reformulation of (1)-(5) is given by

$$\begin{cases} \dot{s} = \left(\sqrt{1 + s_x^2} + \delta\frac{s_{xx}}{1+s_x^2}\right)\Psi[s], & \text{in } [0,1] \times (0,\infty) \\ s(0, \cdot) = 0, & \text{in } [0,1] \\ \Psi(s(0, \cdot)) = g, & \text{in } [0,1] \end{cases} \quad (21)$$

where $\Psi[s] = \big(\mathrm{id} + \mathcal{I}_s\big)^{-1}\mathcal{R}_3(g, s, \dot{s})$. Using (21) and periodic pseudo-differential operators we are able to formulate and prove the following theorem.

**Theorem 4.1.** *(Local Existence) Let $g \in H_p^2([0,1])$ be such that $g > 0$ and $\delta$ a positive real number. There exists $T = T(s_0) > 0$ and strict solution*

$$s \in C^\alpha\big([0,T]; H_p^2([0,1])\big) \cap C^{1+\alpha}\big([0,T]; L_{2,p}([0,1])\big)$$

*to (21). Furthermore, let $s_{g_1}$ and $s_{g_2}$ be solutions to (21) with $H_p^2([0,1])$ boundary data $g_1$ and $g_2$ respectively, then there exist $C, r > 0$ such that if $\|g_1 - g_2\|_{H_p^2([0,1])} \leq r$, then*

$$\|s_{g_1} - s_{g_1}\|_{C^\alpha([0,T]; H_p^2([0,1]))} + \|\dot{s}_{g_1} - \dot{s}_{g_2}\|_{C^\alpha([0,T]; L_{2,p}([0,1]))} \leq C\|g_1 - g_2\|_{H_p^2([0,1])} \quad (22)$$

The remainder of this paper deals with the proof of Theorem 4.1.

5. **Pseudo-differential operators.** Let $m \in \mathbb{R}$ then $S_p^m = S^m([0,1] \times [0,1] \times \mathbb{Z})$ denotes the set of those functions

$$\sigma : [0,1] \times [0,1] \times \mathbb{Z} \to \mathbb{C}$$

that are $C^\infty$ in the first two arguments and for which

$$\left| \triangle_k^\alpha \partial_x^\beta \partial_y^\gamma \sigma(x,y,k) \right| \le C_{\alpha,\beta,\gamma} \langle k \rangle^{m-|\alpha|}$$

for every $x, y \in [0,1]$ and every $\alpha, \beta, \gamma \in \mathbb{N}$. Where for $\sigma \in S_p^m$

$$\langle k \rangle = (1 + |k|^2)^{\frac{1}{2}}$$
$$\triangle_k \sigma(x,y,k) = \sigma(x,y,k+1) - \sigma(x,y,k)$$
$$\triangle_k^\alpha \sigma(x,y,k) = \sum_{\beta \le \alpha} (-1)^{|\alpha - \beta|} \binom{\alpha}{\beta} \sigma(x,y,k+\beta)$$

For $\sigma \in S_p^m$ one can define the operator

$$Op(\sigma)[f](x) = \int_0^1 \sum_{k \in \mathbb{Z}} \sigma(x,y,k) e^{2\pi i k(x-y)} f(y) dy$$

and prove

$$Op(\sigma) \in \mathcal{L}\big(H_p^q([0,1]), L_{2,p}([0,1])\big)$$

for $q \in \mathbb{N}$ such that $m+1 < q$ (see [14]). The symbol class $S_p^m$ is very restrictive due to the $C^\infty$ constraints and we need to define a new class of symbols with the goal of preserving $L_q([0,1])$ boundedness. Kumano-Go and Nagase in [11] and [13] tackled this situation for non-regular symbols on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, so we modify the symbol class given by Kumano-Go and Nagase to apply to operators on the unit circle.

**Definition 5.1.** For $\tau \ge 0$, $S_{\rho,\delta;\tau}^m$ is defined as the set of symbols $\sigma(x,y,k)$ which have continuous derivatives satisfying

$$|\triangle_k^\alpha \partial_x^\beta \partial_y^{\beta'} \sigma| \le C \langle k \rangle^{m+\delta|\beta+\beta'|-\rho|\alpha|} \tag{23}$$

for any $\alpha$ and $|\beta + \beta'| \le [\tau]$ and

$$|\triangle_k^\alpha \partial_x^\beta \partial_y^{\beta'} \sigma(x,y,k) - \triangle_k^\alpha \partial_x^\beta \partial_y^{\beta'} \sigma(x',y',k)| \le C(|x-x'|^{\dot{\tau}} + |y-y'|^{\dot{\tau}}) \langle k \rangle^{m+\delta\tau-\rho\alpha} \tag{24}$$

for $|\beta + \beta'| = [\tau], \tau > 0$ and $|x - x'| \le 1, |y - y'| \le 1$ where $C$ is a constant depending on $\alpha, \beta$ and $\beta'$, and $\dot{\tau} = \tau - [\tau]$

Kumano-Go and Nagase were able to prove several theorems regarding $L_q(\mathbb{R}^n)$-boundedness of pseudo-differential operators with non-regular symbols when $q$ real and larger than one. In particular, they showed that for a symbol

$$\sigma(x,y,k) \in S_{1,\delta';\tau}^0$$

the operator $Op(\sigma)$ is a bounded operator from $L_q$ to $L_q$, provided $0 \le \delta' < 1$ and $0 < \tau \le 1$. Since we restrict ourselves to the $L_{2,p}([0,1])$ context we only need to check conditions (23)-(24) for $|\alpha| = 0, 1, 2$ (see [11]).

Recall our logarithmic kernel, $G$, defined by

$$G(x,y) = \frac{1}{2\pi} \log |1 + e^{-4\pi y} - 2\cos(2\pi x) e^{-2\pi y}|$$

for $x \in [0,1]$ and $y$ nonnegative. Upon differentiating $G$ away from the origin one computes

$$\nabla G(x,y) = \Big( \frac{4\cos(\pi x)\sin(\pi x)}{2\cosh(2\pi y) - 2\cos(2\pi x)}, \frac{2 - 2e^{-2\pi y}\cos(2\pi x)}{1 + e^{-4\pi y} - 2\cos(2\pi x)e^{-2\pi y}} - 2 \Big)$$

and as $y \to 0^{\pm}$ (see [9])

$$\nabla G(x,0) = \Big( v.p.\frac{\cos(\pi x)}{\sin(\pi x)}, \pm\delta - 1 \Big)$$

For negative argument it is easy to see that

$$G(x,-y) = 2y + G(x,y), \quad y > 0$$

This reflection property will be very useful later when we allow $y$ to vary, i.e. when we evaluate, $G$ on $\Gamma_t$. In [9], Guidotti viewed the operators

$$T_G^y[f](x) = \int_0^1 G(x - \tilde{x}, y)f(\tilde{x})d\tilde{x}$$

$$T_K^y[f](x) = \int_0^1 G_y(x - \tilde{x}, y)f(\tilde{x})d\tilde{x}$$

as pseudo-differential operators first for fixed positive $y$ where he computed their symbol as $a_G(k) = \frac{e^{-2\pi|k|y}}{2\pi|k|}$, $k \in \mathbb{Z}^*$, and $a_K(k) = e^{-2\pi|k|y}$, $k \in \mathbb{Z}$ respectively. In other words,

$$T_G^y[f] = Op(a_G)[f]$$

$$T_K^y[f] = Op(a_K)[f]$$

Notice that the symbol $a_G$ does not contain any information about the $k = 0$ mode, so in [9] Guidotti always restricted the context to the case when $T_G$ acts on mean-zero functions. Using Plancherel it is obvious that

$$T_G^y : \tilde{H}_p^m \to \tilde{H}_p^{m+1}$$

$$T_K^y : H_p^m \to H_p^m$$

where $\tilde{H}_p^m = \{v \in H_p^m([0,1]) | \int_0^1 v(x)dx = 0\}$. A straightforward argument shows that $T_G^y$ is well-defined for any $H_p^m$ function and satisfies the same mapping property as in the mean-zero case.

Next consider the operator

$$T_H^y[f](x) = \int_0^1 G_x(x - \tilde{x}, y)f(\tilde{x})d\tilde{x}$$

where we can define its symbol, $a_H$, as follows

$$a_H(k) = -i\,\text{sgn}(k)e^{-2\pi|k|y}$$

When $y$ is positive it is clear that $T_H^y$ maps $H_p^m$ into $H_p^m$, and $T_H^0$ is nothing else but the Hilbert Transform on $[0,1]$ which maps $H_p^m$ into $H_p^m$.

The interesting and more challenging problem is how to define the symbol when $y$ varies. In the following definition we summarize the results proved in [9]

**Definition 5.2.** (Theorem 8 in [9]) Let $s \in H_p^2([0,1])$.

(a) Then
$$\int_0^1 G_y\big(x - \tilde{x}, s(x) - s(\tilde{x})\big) f(\tilde{x}) = Op(a_K)[f](x)$$

for any $f \in H_p^1([0,1])$, where $a_K = a_K(k, x, \tilde{x})$ is given by

$$a_K(k, x, \tilde{x}) = \begin{cases} \exp\Big(-2\pi|k|\big(s(x) - s(\tilde{x})\big)\Big), & k \in \mathbb{Z}, \ s(x) - s(\tilde{x}) \geq 0 \\[2em] -2\delta(k) - \exp\Big(2\pi|k|\big(s(x) - s(\tilde{x})\big)\Big), & k \in \mathbb{Z}, \ s(x) - s(\tilde{x}) < 0 \end{cases}$$

(b) Then
$$\int_0^1 G\big(x - \tilde{x}, s(x) - s(\tilde{x})\big) f(\tilde{x}) d\tilde{x} = Op(a_G)[f](x)$$

for any $f \in L_{2,p}([0,1])$, where $a_G = a_G(k, x, \tilde{x})$ is given by

$$a_G(k, x, \tilde{x}) = \begin{cases} \dfrac{\exp\big(-2\pi|k|[s(x) - s(\tilde{x})]\big)}{2\pi|k|}, & s(x) - s(\tilde{x}) \geq 0 \\[2em] \dfrac{\exp\big(2\pi|k|[s(x) - s(\tilde{x})]\big)}{2\pi|k|} + e^{2\pi i k \tilde{x}} \hat{c}(k, \tilde{x}), & s(x) - s(\tilde{x}) < 0 \end{cases}$$

for $k \in \mathbb{Z}^*$ and where the correction term $\hat{c}$ is given by

$$\hat{c}(\cdot, \tilde{x}) = \mathcal{F}_{x \mapsto k}[-2\big(s(\cdot) - s(\tilde{x})\big) \chi_{[s(\cdot) - s(\tilde{x}) < 0]}]$$

(c) Then
$$\int_0^1 G_x\big(x - \tilde{x}, s(x) - s(\tilde{x})\big) f(\tilde{x}) d\tilde{x} = Op(a_H)[f](x)$$

for any $f \in H_p^1([0,1])$, where $a_H = a_H(k, x, \tilde{x})$ is given by

$$a_H(k, x, \tilde{x}) = \begin{cases} -i\,\mathrm{sgn}(k) \exp\big(-2\pi|k|[s(x) - s(\tilde{x})]\big), & k \in \mathbb{Z}, \ s(x) - s(\tilde{x}) \geq 0 \\[1em] -i\,\mathrm{sgn}(k) \exp\big(2\pi|k|[s(x) - s(\tilde{x})]\big), & k \in \mathbb{Z}, \ s(x) - s(\tilde{x}) < 0 \end{cases}$$

Using the pseudo-differential operator perspective in Definition 5.2 we can derive many properties of the operators in the BIF.

**Proposition 5.3.**
$$Op(a_K) \in \mathcal{L}\big(H_p^1([0,1]), L_{2,p}([0,1])\big)$$
$$Op(a_G) \in \mathcal{L}\big(L_{2,p}([0,1]), L_{2,p}([0,1])\big)$$
$$Op(a_H) \in \mathcal{L}\big(H_p^1([0,1]), L_{2,p}([0,1])\big)$$

We present a technical lemma before proving Proposition 5.3.

**Lemma 5.4.** *Let* $s \in H_p^2([0,1])$ *and* $p(x, y, k) = e^{-2\pi|k||s(x) - s(y)|}$ *for* $k \in \mathbb{Z}^*$, *then for all* $\eta \in (0,1)$ *and* $\alpha = 0, 1, 2$ *there exists a nonnegative constant,* $C_\eta$, *such that*

$$|\triangle_k^\alpha p(x, y, k) - \triangle_k^\alpha p(x', y', k)| \leq C_\eta \left(|x - x'|^\eta + |y - y'|^\eta\right) \langle k \rangle^\eta$$

*Proof.* First consider the function, $f_{k,\eta} : [0, \infty) \to \mathbb{R}$ defined as follows

$$f_{k,\eta}(R) = \begin{cases} 0, & R = 0 \\[1em] \dfrac{1 - \exp\big(-2\pi|k|R\big)}{|k|^\eta R^\eta}, & R > 0 \end{cases} \quad k \in \mathbb{Z}^+, \eta \in (0,1)$$

then there exists a $K_\eta$ such that

$$\sup_{k\in\mathbb{Z}^+,R\geq 0}|f_{k,\eta}(R)|\leq K_\eta$$

Let $x,y,x',y'\in[0,1]$, $\eta\in(0,1)$ and $s\in H_p^2([0,1])$. Suppose without loss of generality $k\in\mathbb{Z}^+$ and

$$|s(x')-s(y')|-|s(x)-s(y)|\geq 0$$

Since $s\in H_p^2([0,1])$, we have $s\in C^1([0,1])$ and

$$\sup_{\substack{x,y,x',y'\in[0,1]\\k\in\mathbb{Z}^*}}\left|\frac{1-\exp\left(-2\pi k\big(|s(x')-s(y')|-|s(x)-s(y)|\big)\right)}{\big[k\big(|s(x')-s(y')|-|s(x)-s(y)|\big)\big]^\eta}\right|\leq K_\eta \qquad (25)$$

Furthermore there exists a constant $C_\eta$ such that

$$|p(x,y,k)-p(x',y',k)|=\left|e^{-2\pi k|s(x)-s(y)|}\left(1-e^{-2\pi k\big(|s(x')-s(y')|-|s(x)-s(y)|\big)}\right)\right|$$

$$\leq K_\eta(|s(x')-s(y')|-|s(x)-s(y)|)^\eta k^\eta e^{-2\pi k|s(x)-s(y)|}$$

$$\leq C_\eta(|x-x'|^\eta+|y-y'|^\eta)\langle k\rangle^\eta$$

Next, notice

$$|\triangle_k p(x,y,k)-\triangle_k p(x',y',k)|$$
$$=|p(x,y,k)-p(x',y',k)-p(x,y,k+1)+p(x',y',k+1)|$$

so that by the previous argument we have that there exists a constant $C_\eta$ such that

$$|\triangle_k p(x,y,k)-\triangle_k p(x',y',k)|\leq C_\eta(|x-x'|^\eta+|y-y'|^\eta)\langle k\rangle^\eta$$

In a similar fashion one can show

$$|\triangle_k^2 p(x,y,k)-\triangle_k^2 p(x',y',k)|\leq C_\eta(|x-x'|^\eta+|y-y'|^\eta)\langle k\rangle^\eta$$

$\square$

*Proof.* (Proof of Proposition 5.3) Clearly the operators are linear in the argument $f$, so we move on to the continuity estimate.

Suppose $s(x)-s(\tilde{x})\geq 0$. We show $Op(a_G)\in\mathcal{L}\big(L_{2,p}([0,1]),L_{2,p}([0,1])\big)$ by checking the conditions of Theorem 2.2 in [11]. We claim $a_G\in S_{1,\frac{1}{2};\frac{1}{2}}^0$ for $\alpha=0,1,2$. Observe that

$$a_G(x,\tilde{x},k)=\frac{p(x,\tilde{x},k)}{2\pi|k|}$$

where $p(x,\tilde{x},k)=e^{-2\pi|k||s(x)-s(\tilde{x})|}$ for $k\in\mathbb{Z}^*$, so that by Lemma 5.4 ($\eta=\frac{1}{4}$) we have $a_G\in S_{1,\frac{1}{2};\frac{1}{2}}^0$ for $\alpha=0,1,2$ and the norm estimate

$$\|Op(a_G)[f]\|_{L_{2,p}}\leq C\|f\|_{L_{2,p}}$$

To finish the case $s(x)-s(\tilde{x})\geq 0$ we view $Op(a_K)$ and $Op(a_H)$ as a composition of two operators. It is clear that

$$Op(c|k|)\in\mathcal{L}\big(H_p^1([0,1]),L_{2,p}([0,1])\big)$$

for all $c \in \mathbb{R}$, so that by the equalities

$$Op(a_K)[f] = Op\left(\frac{p(\cdot, \tilde{x}, k)}{2\pi|k|}\right)Op(2\pi|k|)[f]$$

$$Op(a_H)[f] = Op\left(\frac{p(\cdot, \tilde{x}, k)}{2\pi|k|}\right)Op\left(-2\pi i|k|\operatorname{sgn}(k)\right)[f]$$

we have

$$\|Op(a_K)[f]\|_{L_{2,p}} \leq C\|f\|_{H_p^1}$$

$$\|Op(a_H)[f]\|_{L_{2,p}} \leq C\|f\|_{H_p^1}$$

Lastly, note for any $x, \tilde{x} \in [0,1)$

$$Op(a_G)[f] = 2\int_0^1 |s(\cdot) - s(\tilde{x})|\chi_{[s(\cdot)-s(\tilde{x})<0]}f(\tilde{x})d\tilde{x} + Op\left(\frac{p(\cdot, \tilde{x}, k)}{2\pi|k|}\right)[f]$$

$$Op(a_H)[f] = Op\left(\frac{p(\cdot, \tilde{x}, k)}{2\pi|k|}\right)Op\left(-2\pi i|k|\operatorname{sgn}(k)\right)[f]$$

and

$$Op(a_K)[f] = Op\left(\frac{p(\cdot, \tilde{x}, k)}{2\pi|k|}\right)Op(2\pi|k|)[f] - 2\int_0^1 \chi_{[s(\cdot)-s(\tilde{x})]<0]}f(\tilde{x})d\tilde{x}$$

$$- 2Op\left(\frac{p(\cdot, \tilde{x}, k)}{2\pi|k|}\right)Op(2\pi|k|)[\chi_{[s(\cdot)-s(\tilde{x})]<0]}f]$$

With these representations of the integral operators, the estimates derived above in the $s(x) - s(\tilde{x}) \geq 0$ case can be applied to complete the proof.    □

It will be beneficial to introduce a few operators closely related to $Op(a_G), Op(a_H)$ and $Op(a_K)$.

**Definition 5.5.** Let $f \in H_p^2([0,1]), s \in H_p^2([0,1])$

(a)

$$DOp_{(s(\cdot)-s(\tilde{x}))}^f(a_G^{+,-}): \quad H_p^2([0,1]) \to L_{2,p}([0,1])$$

$$DOp_{(s(\cdot)-s(\tilde{x}))}^f(a_G^+)[r] = \int_0^1 \sum_{k \in \mathbb{Z}^*} -e^{-2\pi|k||[s(\cdot)-s(\tilde{x})]|}e^{2\pi ik(\cdot-\tilde{x})}[r(\cdot) - r(\tilde{x})]f(\tilde{x})d\tilde{x}$$

$$\text{if } s(\cdot) - s(\tilde{x}) \geq 0$$

$$DOp_{(s(\cdot)-s(\tilde{x}))}^f(a_G^-)[r] = \int_0^1 \sum_{k \in \mathbb{Z}^*} e^{2\pi|k||[s(\cdot)-s(\tilde{x})]|}e^{2\pi ik(\cdot-\tilde{x})}[r(\cdot) - r(\tilde{x})]f(\tilde{x})d\tilde{x}$$

$$+ \int_0^1 \sum_{k \in \mathbb{Z}^*} e^{2\pi ik\tilde{x}}\mathcal{F}_{\cdot \mapsto k}[-2(r(\cdot) - r(\tilde{x}))\chi_{[s(\cdot)-s(\tilde{x})<0]}]f(\tilde{x})d\tilde{x}$$

$$\text{if } s(\cdot) - s(\tilde{x}) < 0$$

(b)

$$DOp^f_{(s(\cdot)-s(\tilde{x}))}(a_K): \quad H^2_p([0,1]) \to L_{2,p}([0,1])$$

$$DOp^f_{(s(\cdot)-s(\tilde{x}))}(a_K)[r] = \int_0^1 \sum_{k \in \mathbb{Z}^*} -2\pi|k|e^{-2\pi|k||s(\cdot)-s(\tilde{x})|}e^{2\pi i k(\cdot-\tilde{x})}[r(\cdot)-r(\tilde{x})]f(\tilde{x})d\tilde{x}$$

(c)

$$DOp^f_{(s(\cdot)-s(\tilde{x}))}(a_H): \quad H^2_p([0,1]) \to L_{2,p}([0,1])$$

$$DOp^f_{(s(\cdot)-s(\tilde{x}))}(a_H)[r] = \int_0^1 \sum_{k \in \mathbb{Z}^*} 2\pi i k e^{-2\pi|k||s(\cdot)-s(\tilde{x})|}e^{2\pi i k(\cdot-\tilde{x})}[r(\cdot)-r(\tilde{x})]f(\tilde{x})d\tilde{x}$$

The operators in Definition 5.5 are the building blocks of the Fréchet derivatives used in the local existence result. The following proposition states the operators mapping properties and its proof is similar to that of Proposition 5.3.

**Proposition 5.6.** *Let* $f \in H^2_p([0,1]), s \in H^2_p([0,1])$

$$DOp^f_{(s(x)-s(\tilde{x}))}(a_K) \in \mathcal{L}\big(H^2_p([0,1]), L_{2,p}\big)$$

$$DOp^f_{(s(x)-s(\tilde{x}))}(a_G^{+,-}) \in \mathcal{L}\big(H^2_p([0,1]), L_{2,p}\big)$$

$$DOp^f_{(s(x)-s(\tilde{x}))}(a_H) \in \mathcal{L}\big(H^2_p([0,1]), L_{2,p}\big)$$

Next we explore the continuity properties of the integral operators in the BIF. The explicit knowledge of the symbols is crucial in the proofs. We begin our investigation of the BIF operators by defining the notation

$$Op_{f(x,\tilde{x})}(a_G), Op_{f(x,\tilde{x})}(a_K), Op_{f(x,\tilde{x})}(a_H)$$

as the Fourier symbols of the integral operators with respective kernels

$$G\big(x-\tilde{x}, f(x,\tilde{x})\big), G_y\big(x-\tilde{x}, f(x,\tilde{x})\big), G_x\big(x-\tilde{x}, f(x,\tilde{x})\big)$$

**Theorem 5.7.** *Let* $f \in L_{2,p}([0,1])$ *and* $v, w \in H^2_p([0,1])$. *Then there exists a constant,* $C$, *such that*

$$\|Op_{v(\cdot)-v(\tilde{x})}(a_G)[f] - Op_{w(\cdot)-w(\tilde{x})}(a_G)[f]\|_{L_{2,p}} \le C\|f\|_{L_{2,p}}\|v-w\|_{H^2_p} \qquad (26)$$

*Proof.* See Appendix. □

**Theorem 5.8.** *Let* $s, (s_n)_{n \in \mathbb{N}} \subset H^2_p([0,1])$, $s_n \ge 0$ *be such that* $s_n \to s$ *in* $H^2_p([0,1])$ *as* $n \to \infty$. *Then there exists a nonnegative constant,* $C$, *such that*

$$\|Op_{s_n(x)}(a_J) - Op_{s(x)}(a_J)\|_{\mathcal{L}(L_{2_p}, L_{2,p})} \le C\|s_n - s\|_{H^2_p} \qquad (27)$$

$$\|Op_{-s_n(\tilde{x})}(a_J) - Op_{-s(\tilde{x})}(a_J)\|_{\mathcal{L}(H^2_p, L_{2,p})} \le C\|s_n - s\|_{H^2_p} \qquad (28)$$

$$\|Op_{s_n(x)-s_n(\tilde{x})}(a_{J'}) - Op_{s(x)-s(\tilde{x})}(a_{J'})\|_{\mathcal{L}(H^2_p, L_{2,p})} \le C\|s_n - s\|_{H^2_p} \qquad (29)$$

*for* $J = G, H, K$ *and* $J' = H, K$.

*Proof.* See Appendix. □

The local existence result needs a similar Lipschitz continuity estimate as in Theorem 5.8 applied to the operators in Definition 5.5. The following theorem involves the same arguments as in the proof of Theorem 5.8 and is stated without proof.

**Theorem 5.9.** *Let $s, (s_n)_{n\in\mathbb{N}} \subset H_p^2([0,1])$, $s_n \geq 0$, $h, (h_n)_{n\in\mathbb{N}} \subset H_p^1([0,1])$, and $f, (f_n)_{n\in\mathbb{N}} \subset L_{2,p}([0,1])$ such that $s_n \to s$ in $H_p^2([0,1])$, $h_n \to h$ in $H_p^1([0,1]$, and $f_n \to f$ in $L_{2,p}([0,1])$ as $n \to \infty$. Then there exists a nonnegative constant, $C$, such that*

$$\|DOp^f_{s_n(x)}(a_J) - DOp^f_{s(x)}(a_J)\|_{\mathcal{L}(H_p^2, L_{2,p})} \leq C\|s_n - s\|_{H_p^2}\|f\|_{L_{2,p}} \quad (30)$$

$$\|DOp^h_{-s_n(\tilde{x})}(a_J) - DOp^h_{-s(\tilde{x})}(a_J)\|_{\mathcal{L}(H_p^2, L_{2,p})} \leq C\|s_n - s\|_{H_p^2} \quad (31)$$

$$\|DOp^{f_n}_{s(x)}(a_G) - DOp^f_{s(x)}(a_G)\|_{\mathcal{L}(H_p^2, L_{2,p})} \leq C\|f_n - f\|_{L_{2,p}} \quad (32)$$

$$\|DOp^{h_n}_{s(x)}(a_{J'}) - DOp^h_{s(x)}(a_{J'})\|_{\mathcal{L}(H_p^2, L_{2,p})} \leq C\|h_n - h\|_{H_p^1} \quad (33)$$

$$\|DOp^h_{s_n(x)-s_n(\tilde{x})}(a_{J'}) - DOp^h_{s(x)-s(\tilde{x})}(a_{J'})\|_{\mathcal{L}(H_p^2, L_{2,p})} \leq C\|s_n - s\|_{H_p^2} \quad (34)$$

$$\|DOp^{h_n}_{s(x)-s(\tilde{x})}(a_{J'}) - DOp^h_{s(x)-s(\tilde{x})}(a_{J'})\|_{\mathcal{L}(H_p^2, L_{2,p})} \leq C\|h_n - h\|_{H_p^1} \quad (35)$$

*for $J = G, H, K$ and $J' = H, K$.*

Next we compute the Fréchet Derivative of the BIF for $\gamma_t u$ in the initial condition. Then we use Theorem 5.7 and Theorem 5.9 to show both the BIF and its Fréchet Derivative are well-defined for small times.

Let $\epsilon, T > 0$, $s, r \in C^\alpha([0,T]; H_p^2) \cap C^{1+\alpha}([0,T]; L_{2,p})$, and $g \in H_p^2([0,1])$. We use (19) to construct an equation that $\frac{\Psi[s+\epsilon r] - \Psi[s]}{\epsilon}$ must satisfy. To slightly simplify the notation in the following let

$$B[s](\tilde{x}) = \mathcal{N}\left[\int_0^1 (-s_z(z), 1) \cdot \nabla G(\tilde{x} - z, -s(z))\Psi[s](z)dz\right]$$

$$I_s[\Psi](x) = \Psi[s](x) + \int_0^1 (-s_x(\tilde{x}), 1)\nabla G(x - \tilde{x}, s(x) - s(\tilde{x}))\Psi[s](\tilde{x})d\tilde{x}$$

$$- \int_0^1 G(x - \tilde{x}, s(x))B[s](\tilde{x})d\tilde{x}$$

$$\triangle_{\epsilon r}\mathcal{A}_s = \frac{\mathcal{A}[s + \epsilon r] - \mathcal{A}[s]}{\epsilon}, \quad \mathcal{A} = \Psi, B$$

Computing $\frac{1}{\epsilon}(I_{s+\epsilon r} - I_s)$ yields

$$\triangle_{\epsilon r}\Psi_s + \frac{1}{\epsilon}\left(Op_{(s+\epsilon r)(\cdot)-(s+\epsilon r)(\tilde{x})}(a_K) - Op_{s(\cdot)-s(\tilde{x})}(a_K)\right)[\Psi[s+\epsilon r](\tilde{x})]$$

$$+ \frac{1}{\epsilon}\left(Op_{(s+\epsilon r)(\cdot)-(s+\epsilon r)(\tilde{x})}(a_H) - Op_{s(\cdot)-s(\tilde{x})}(a_H)\right)[-s_x(\tilde{x})\Psi[s+\epsilon r](\tilde{x})]$$

$$+ Op_{s(\cdot)-s(\tilde{x})}(a_K)[\triangle_{\epsilon r}\Psi_s(\tilde{x})] + Op_{s(\cdot)-s(\tilde{x})}(a_H)[-s_x(\tilde{x})\triangle_{\epsilon r}\Psi_s(\tilde{x})]$$

$$- \frac{1}{\epsilon}\left(Op_{(s+\epsilon r)(\cdot)}(a_G) - Op_{s(\cdot)}(a_G)\right)[B[s+\epsilon r](\tilde{x})] - Op_{s(\cdot)}(a_G)[\triangle_{\epsilon r}B_s(\tilde{x})]$$

$$+ Op_{(s+\epsilon r)(\cdot)-(s+\epsilon r)(\tilde{x})}(a_H)[-r_x(\tilde{x})\Psi[s+\epsilon r](\tilde{x})] \quad (36)$$

To let $\epsilon$ go to zero we use the following proposition.

**Proposition 5.10.** *Let $\epsilon > 0$ and $f \in H_p^1([0,1])$. Assume either*

$$s \in H^+ := \{v \in H_p^2([0,1]) : v \geq 0\}$$

*and $r \in H^0 := \{v \in H_p^2([0,1]) : \int v = 0\}$ or $s = s_0 \equiv 0$ and $r \in H^+$ then*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(Op_{(s+\epsilon r)(x)-(s+\epsilon r)(\tilde{x})}(a_J) - Op_{s(x)-s(\tilde{x})}(a_J)\big)[f](x) = DOp_{s(x)-s(\tilde{x})}^f(a_J)[r](x)$$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(Op_{\pm(s+\epsilon r)(x)}(a_{J'}) - Op_{\pm s(x)}(a_{J'})\big)[f](x) = DOp_{\pm s(x)}^f(a_{J'})[r](x)$$

*for $J = G, H, K$ and $J' = G, K$.*

If we define

$$D\Psi_s = \lim_{\epsilon \to 0} \frac{\Psi[\cdot + \epsilon r] - \Psi[\cdot]}{\epsilon}$$

then using Proposition 5.10 letting $\epsilon \to 0$ in (36) yields

$$D\Psi_s + \int_0^1 \big(-s_x(\tilde{x}), 1\big) \cdot \nabla G\big(\cdot - \tilde{x}, s(\cdot) - s(\tilde{x})\big) D\Psi_s(\tilde{x})d\tilde{x}$$

$$- \int_0^1 G\big(\cdot - \tilde{x}, s(\cdot)\big)\mathcal{N}\left[\int_0^1 \big(-s_z(z), 1\big) \cdot \nabla G\big(\tilde{x} - z, -s(z)\big) D\Psi_s(z)dz\right] d\tilde{x} + DI_s[r] \tag{37}$$

where

$$DI_s[r](x) = DOp_{(s(x)-s(\tilde{x}))}^{\Psi[s]}(a_K)[r](x) + DOp_{(s(x)-s(\tilde{x}))}^{-s_x\Psi[s]}(a_H)[r](x)$$

$$- DOp_{s(x)}^{B[s]}(a_G)[r](x) + Op_{s(x)-s(\tilde{x})}(a_H)[-r_x\Psi[s]](x)$$

$$- Op_{s(x)}(a_G)\left[\mathcal{N}\big[DOp_{-s(z)}^{\Psi[s]}(a_K)[r](\tilde{x}) + DOp_{-s(z)}^{-s_x\Psi[s]}(a_H)[r](\tilde{x})\right.$$

$$\left.+ Op_{-s(z)}(a_H)[-r_z\Psi[s]](\tilde{x})\big]\right](x)$$

The first three terms of (37) should look familiar since they correspond to the operator $I_s$ applied to $D\Psi_s$. To finish deriving the equation $D\Psi_s$ must satisfy, we compute

$$\frac{\mathcal{R}_3(g, s + \epsilon r, \dot{s} + \epsilon r) - \mathcal{R}_3(g, s, \dot{s})}{\epsilon}$$

and let $\epsilon$ go to zero. By Proposition 5.10 we have

$$\lim_{\epsilon \to 0} \frac{\mathcal{R}_3(g, s + \epsilon r, \dot{s} + \epsilon r) - \mathcal{R}_3(g, s, \dot{s})}{\epsilon} = D\mathcal{R}^1[r] + D\mathcal{R}^2[\dot{r}] \tag{38}$$

where

$$D\mathcal{R}^1[r](x) = DOp_{s(x)}^g(a_K)[r](x) + DOp_{s(x)}^{\mathcal{N}(\mathcal{R}_1(g,s,\dot{s}))}(a_G)[r](x)$$

$$+ Op_{s(x)}(a_G)\big[\mathcal{N}[DOp_{-s(z)}^{\dot{s}(z)}(a_G)[r](\tilde{x})]\big](x)$$

$$+ \left(DOp_{s(x)}^{\dot{s}}(a_G) - DOp_{s(x)-s(\tilde{x})}^{\dot{s}}(a_G)\right)[r](x)$$

$$D\mathcal{R}^2[\dot{r}](x) = \left(Op_{s(x)}(a_G) - Op_{s(x)-s(\tilde{x})}(a_G)\right)[\dot{r}](x)$$

$$+ Op_{s(x)}(a_G)\left[\mathcal{N}\big[\left(Op_{-s(z)}(a_G) - Op_0(a_G)\right)[\dot{r}](\tilde{x})\big]\right](x)$$

Combining (37) and (38) we have the following equation for $D\Psi_s$

$$I_s D\Psi = \big(D\mathcal{R}^1 - DI_s\big)[r] + D\mathcal{R}^2[\dot{r}] \tag{39}$$

which formally for the moment gives

$$D\Psi[r, \dot{r}] = I_s^{-1}\left[\left(D\mathcal{R}^1 - DI_s\right)\right][r] + I_s^{-1}\left[D\mathcal{R}^2\right][\dot{r}] =: D\Psi_s^1[r] + D\Psi_s^2[\dot{r}] \qquad (40)$$

Now we are in a position to understand the setting in which the BIF and its Fréchet derivative are well-defined.

**Theorem 5.11.** *Let $T > 0$, $r \in C^{1+\alpha}\big([0, T]; L_{2,p}\big) \cap C^\alpha\big([0, T], H^+\big)$ and define*

$$Y^+ = \{f : f(t, \cdot) \in H^+, \dot{f}(t, \cdot) \in L_{2,p}^+, \forall t \in [0, T]\}$$

*If $g \in H_p^2([0, 1])$ is such that $g > 0$, $s(0, \cdot) = s_0 \equiv 0$ and $\dot{s}(0, \cdot) = g$ then*

$$D\Psi_s(s_0) : Y^+ \to L_{2,p}$$

$$D\Psi_s(s_0)[r] = gr - \int_0^1 g(\tilde{x})r(\tilde{x})d\tilde{x} + \mathcal{N}\big[\mathcal{N}[g]\big]r$$

*is a linear operator and there exists an $\eta > 0$ such that, for each $s \in \mathbb{B}_{H^+}(s_0, \eta)$, $D\Psi_s$ is given by* (40).

*Proof.* Let $s = s_0$, $\dot{s} = g$, $r \in Y^+$ and recall

$$\mathcal{F}\int_0^1 G(\cdot - \tilde{x}, 0)\mathcal{N}[f]d\tilde{x} = \mathcal{F}[f], \qquad (41)$$

In this case, evaluating $D\mathcal{R}^2[\dot{r}]$ or any operator with $s_x$ yields the zero operator. Furthermore, a straightforward calculation yields

$$DOp_{s(x)-s(\tilde{x})}^{\Psi[s]}(a_K)[r](x)\big|_{s=s_0} = r(x)\int_0^1 \sum_{k\in\mathbb{Z}^*}(-2\pi|k|)e^{2\pi ik(x-\tilde{x})}g(\tilde{x})d\tilde{x}$$

$$-\int_0^1 \sum_{k\in\mathbb{Z}^*}(-2\pi|k|)e^{2\pi ik(x-\tilde{x})}r(\tilde{x})g(\tilde{x})d\tilde{x}$$

$$= -r(x)\mathcal{N}[g](x) + \mathcal{N}[rg](x)$$

$$Op_{s(x)-s(\tilde{x})}(a_H)[-r_x\Psi[s]](x)\big|_{s=s_0} = \big(\mathcal{F}^{-1}(-i\operatorname{sgn}(k))\mathcal{F}(-r_x g)\big)(x)$$

$$DOp_{-s(z)}^{\Psi[s]}(a_K)[r](\tilde{x})\big|_{s=s_0} = \mathcal{N}[rg](\tilde{x})$$

$$DOp_{s(x)}^{B[s]}(a_G)[r](x)\big|_{s=s_0} = r(x)\int_0^1 \sum_{k\in\mathbb{Z}^*}(-2\pi|k|)e^{2\pi ik(x-\tilde{x})}B[s_0](\tilde{x})d\tilde{x}$$

$$= r(x)\mathcal{N}\big[\mathcal{N}[g]\big](x)$$

$$Op_{-s(z)}(a_H)[-r_z\Psi[s]](\tilde{x})\big|_{s=s_0} = \big(\mathcal{F}^{-1}(-i\operatorname{sgn}(k))\mathcal{F}(-r_z g)\big)(\tilde{x})$$

Using (41) along with the definition of $DI_s[r]$ we compute

$$DI_s[r]\big|_{s=s_0} = -r\mathcal{N}[g] - r\mathcal{N}\big[\mathcal{N}[g]\big] \qquad (42)$$

To calculate $D\mathcal{R}^1[r]$, first notice

$$\mathcal{R}_1(g, s, \dot{s})\big|_{(s=s_0, \dot{s}=g)} = \int_0^1 g(\tilde{x})d\tilde{x}$$

so that

$$\mathcal{N}[\mathcal{R}_1(g, s, \dot{s})\big|_{(s=s_0, \dot{s}=g)}] \equiv 0$$

Thus all the terms in $D\mathcal{R}^1$ cancel except $DOp^g_{s(x)}(a_K)[r]$ and

$$Op_{s(x)}(a_G)\big[\mathcal{N}[DOp^{\dot{s}(z)}_{-s(z)}(a_G)[r](\tilde{x})]\big](x)$$

for which one can discover

$$D\mathcal{R}^1[r] = -r\mathcal{N}[g] + gr - \int_0^1 g(\tilde{x})r(\tilde{x})d\tilde{x} \tag{43}$$

Combining (39) , (42) and (43) we have

$$I_s D\Psi_s[r,\dot{r}]\big|_{(s=s_0,\dot{s}=g)} = gr - \int_0^1 g(\tilde{x})r(\tilde{x})d\tilde{x} + r\mathcal{N}\big[\mathcal{N}[g]\big] + D\mathcal{R}^2[\dot{r}]$$

$$= gr - \int_0^1 g(\tilde{x})r(\tilde{x})d\tilde{x} + \mathcal{N}\big[\mathcal{N}[g]\big]r + 0[\dot{r}] \tag{44}$$

Furthermore, $I_s\big|_{s=s_0} = \mathrm{id}$, so that (44) reduces to

$$D\Psi_s[r]\big|_{(s=s_0,\dot{s}=g)} = gr - \int_0^1 g(\tilde{x})r(\tilde{x})d\tilde{x} + \mathcal{N}\big[\mathcal{N}[g]\big]r$$

Clearly $D\Psi_s$ is linear in $r$ and using Theorem 5.7 there exists $\eta > 0$ such that $I_s^{-1}$ exists provided $s \in \mathbb{B}_{H^+}(s_0,\eta)$, so that $D\Psi_s$ is given by (40) in a $\eta$-neighborhood of $s_0$. $\qquad\square$

**Corollary 5.12.** *Let $T > 0$, $h \in H^+$, and $s \in Y^+$ and view $\Psi[s]$ as dependent on the parameter, $g$. Then*

$$D\Psi_g : H^+ \to L_{2,p}([0,1])$$

$$D\Psi_g(s,g)[h](x) = I_s^{-1}\left[\int_0^1 G_y\big(x-\tilde{x}, s(x)\big)h(\tilde{x})d\tilde{x}\right]$$

*in a neighborhood of $s = s_0$.*

**Remark 5.13.** Theorem 5.11 shows that $(\mathrm{id} + \mathcal{I}_s)$ is invertible in a $H^+$ neighborhood of $s_0$ which proves the BIF (21) is well-defined.

6. **Local existence.** We have shown that one can write the concentration on the free-boundary as an operator depending solely on the free-boundary. Furthermore the Fréchet derivative of that operator is well-defined. Now we view (21) as a dynamical system and exploit maximal regularity to obtain local existence and continuous dependence of the boundary datum.

**Lemma 6.1.** *Let $\delta > 0, g \in H_p^2([0,1])$ be such that $g > 0$ and consider the operator*

$$F : H_p^2([0,1]) \subset L_{2,p}([0,1]) \to L_{2,p}([0,1])$$

$$F[v] = \left(\sqrt{1+v_x^2} + \delta\frac{v_{xx}}{1+v_x^2}\right)\Psi[v]$$

*Then $F$ is Fréchet differentiable at $v = s_0$ and $F'(s_0)$ is sectorial in $L_{2,p}([0,1])$.*

*Proof.* From Theorem 5.11, Theorem 5.9 and direct computation we have

$$F'(s_0)v = \big(\delta g\triangle + D\Psi_s(s_0)\big)v$$

The Laplacian is a sectorial operator in $L_{2,p}$ with domain of definition $H_p^2$. In addition, under the appropriate regularity assumptions non-constant coefficient operators are sectorial whenever their constant coefficient counterpart is (see [3]). For $\delta g \in H_p^2([0,1])$, $\delta g \triangle$ is sectorial in $L_{2,p}([0,1])$.

Next, view the operator $D\Psi_s(s_0)$ as a perturbation of $\delta g \triangle$ and refer to Proposition 2.4.1 in [12]. To summarize the proposition, we need a space $X_\alpha$ belonging to the class $J_\alpha$ between $H_p^2([0,1])$ and $L_{2,p}([0,1])$ for which

$$D\Psi_s(s_0) \in \mathcal{L}\big(X_\alpha, L_{2,p}([0,1])\big)$$

$\alpha \in [0,1)$. Under these conditions $\delta g \triangle + D\Psi_{s_0}$ will be sectorial in $L_{2,p}([0,1])$. We claim $X_{\frac{1}{2}} := H_p^1([0,1])$ satisfies the assumptions of the proposition. Obviously $H_p^1([0,1])$ is in between the domain, $H_p^2([0,1])$, and $L_{2,p}([0,1])$, so we move on to proving that it is class $J_{\frac{1}{2}}$. To realize the norm estimate needed in the definition of class $J_{\frac{1}{2}}$, we apply Theorem 4.17 in [1]. There exists a nonnegative constant $c$ such that

$$\|w\|_{H_p^1([0,1])} \leq c\|w\|_{L_{2,p}([0,1])}^{\frac{1}{2}} \|w\|_{H_p^2([0,1])}^{\frac{1}{2}}, \quad \forall w \in H_p^2([0,1])$$

By Theorem 5.11, $D\Psi_s(s_0) \in \mathcal{L}\big(H_p^1([0,1]), L_{2,p}([0,1])\big)$, and we complete the proof. $\qquad\square$

A final continuity estimate is needed to prove Theorem 4.1.

**Lemma 6.2.** *There exist a $R_0, L > 0$ such that the operator*

$$S : H_p^2([0,1]) \to L_{2,p}([0,1])$$
$$s \mapsto \sqrt{1 + s_x^2} + \delta \frac{s_{xx}}{1 + s_x^2}$$

*satisfies*

$$\|DS_s - DS_r\|_{\mathcal{L}(H_p^2([0,1]), L_{2,p}([0,1]))} \leq L\|s - r\|_{H_p^2([0,1])}$$

*for $s, r \in \mathbb{B}_{H_p^2([0,1])}(0, R_0)$ where $DS_r$ is the Fréchet derivative of $S$ evaluated at $r$.*

*Proof.* From results in Chapter 8 of [12] it is enough to show differentiability with respect to the arguments of the real function $(x, s, p, q) \mapsto S(x, s, p, q) = \sqrt{1 + p^2} + \delta \frac{q}{1+p^2}$. We determine $R_0$ by checking twice continuous differentiability. Upon differentiating with respect to $p$ and $q$ we see that the real function $F$ is $C^2(\mathbb{R}, \mathbb{R})$ in both variables and thus we have the result. $\qquad\square$

**Proof of Theorem 4.1**

*Proof.* In [12], Lunardi gave an abstract local existence result. We place (21) within that framework in the following manner. Let $T > 0$ and define the operator

$$F : [0, T] \times H^+ \to L_{2,p}([0,1])$$
$$(t, s) \to \big(\sqrt{1 + s_x^2} + \delta \frac{s_{xx}}{1 + s_x^2}\big)\Psi[s]$$

Then $A = F'(s_0)$ is a sectorial operator from $H^+$ to $L_{2,p}([0,1])$ by Lemma 6.1 and $F(0, s_0) = g \in D_A(\alpha, \infty)$. Furthermore, by Theorems 5.7, 5.9 and 5.11 and

noting the operators $F, \Psi, D\Psi$, and $DF$ are independent of time we have there exist $R, L, K > 0$ such that

$$\|D\Psi^1_{(t,v)} - D\Psi^1_{(t,w)}\|_{\mathcal{L}(H^2_p, L_{2,p})} \leq L\|v - w\|_{H^2_p([0,1])} \tag{45}$$

$$\|D\Psi^2_{(t,v)} - D\Psi^2_{(t,w)}\|_{\mathcal{L}(L_{2,p}, L_{2,p})} \leq L\|v - w\|_{H^2_p([0,1])} \tag{46}$$

$$\|\Psi(t,s) - \Psi(\bar{t},s)\|_{L_{2,p}([0,1])} \leq K|t - \bar{t}|^\alpha \tag{47}$$

and

$$\|D\Psi^1_{(t,s)} + D\Psi^1_{(\bar{t},s)}\|_{\mathcal{L}(H^2_p, L_{2,p})} + \|D\Psi^2_{(t,s)} + D\Psi^2_{(\bar{t},s)}\|_{\mathcal{L}(H^2_p, L_{2,p})} \leq K|t - \bar{t}|^\alpha \tag{48}$$

for all $t, \bar{t} \in [0,T], v, w \in \mathbb{B}_{H^2_p}(s_0, R) \subset L_{2,p}$. Therefore, by Lemma 6.1, Lemma 6.2, and (45)-(48)

$$\dot{s} = F[s]$$
$$s(0) = 0$$

satisfies all conditions of Theorem 8.1.3 in [12]. The slight modification of Lunardi's proof is as follows. Given $v \in C^\alpha([0,T]; H^+) \cap C^{1+\alpha}([0,T]; L_{2,p}([0,1]))$, away from the initial condition $DF$ depends linearly on $v$ and $\dot{v}$. Thus instead of only using the fact that $v \in C^\alpha([0,\delta]; H^+)$, we use $v \in C^{1+\alpha}([0,\delta]; L_{2,p}([0,1]))$. Define

$$\|v\|_Y := \|v\|_{C^\alpha([0,T]; H^2_p([0,1]))} + \|\dot{v}\|_{C^\alpha([0,T]; L_2([0,1]))}$$

and set

$$Y_0 =$$
$$\{s \in C^\alpha([0,T]; H^+) \cap C^{1+\alpha}([0,T]; L_{2,p}) : (s, \dot{s})|_{t=0} = (0, g), \|s\|_{C^\alpha([0,T]; H^2_p)} \leq \rho_0\}$$

where $\rho_0$ is the same as in the proof given in [12]. Define the operator $\Gamma$ in $Y_0$, by $\Gamma(v) = s$, where $s$ is the solution of

$$\dot{s}(t) = As(t) + [F(t, v(t)) - Av(t)], \quad 0 \leq t \leq T, \quad s(0, \cdot) = 0$$

By Theorem 4.3.1 in [12] and the compatibility condition $F(0, s_0) \in D_A(\alpha, \infty)$, $\Gamma$ maps $Y_0$ into $C^\alpha([0,T]; H^+) \cap C^{1+\alpha}([0,T]; L_2([0,1]))$, and since we linearize about $(0, s_0)$, $\Gamma$ maps into $Y_0$ by Theorem 8.1.3 in [12] . To see that $\Gamma$ is a contractive map, let $v, w \in Y_0$, then by Theorem 4.3.2. in [12] there exists $C > 0$ such that

$$\|\Gamma(v) - \Gamma(w)\|_Y \leq C\|[F(v) - F(w) - A(v - w)]\|_{C^\alpha([0,T], L_2)}$$

and by the fact

$$F(t, v(t)) - F(t, w(t)) - A(v(t) - w(t)) = \int_0^1 \left[DF_{(t, \sigma v(t) + (1-\sigma)w(t))} - A\right] d\sigma(v(t) - w(t))$$

we realize

$$\|\Gamma(v) - \Gamma(w)\|_Y \leq \mathcal{C}T^\alpha(\|v - w\|_{C^\alpha([0,T]; H^2)} + \|\dot{v} - \dot{w}\|_{C^\alpha([0,T]; L_2)}) \tag{49}$$

$$\leq \mathcal{C}T^\alpha\|v - w\|_Y \tag{50}$$

where $\mathcal{C} = \mathcal{C}(L, K, \rho_0, \|D\Psi^2_s\|_{\mathcal{L}(L_2, L_2)}, \|D\Psi^1_s\|_{\mathcal{L}(H_2, L_2)})$ (See [12] for the details on estimates). Thus for $T$ small, $\Gamma$ is a contractive self-map.

Upon inspection of the definition of $\Psi$ one sees it depends on the parameter $g$, the boundary datum, so we apply dependence on parameters results in [12]. Let

$$F : [0, T] \times H^+ \times \{g \in H_p^2([0, 1]) : g > 0\} \to L_{2,p}([0, 1])$$

$$(t, s, g) \to \left(\sqrt{1 + s_x^2} + \delta \frac{s_{xx}}{1 + s_x^2}\right) \Psi[s, g]$$

By Theorem 5.7, Theorem 5.9, Corollary 5.12, and Lemma 6.1 $F$ satisfies all the assumptions of Theorem 8.3.2 in [12] and $s$ depends continuously on the boundary datum with norm estimate (22). $\qquad\square$

7. **Concluding remarks.** Utilizing a boundary integral formulation has proven useful in capturing the interaction between the concentration of solute and sharp interface. The formulation led to pseudo-differential operators with symbols that could be explicitly manipulated. Under the crucial assumption that $\delta$ be positive, we were able to use maximal regularity arguments to prove local existence and continuous dependence on the boundary datum for the sharp interface.

Recall the motivation of our analysis is to prove stability of the planar solutions. The local in time result derived here will be essential in the following manner. First, the sharp interface needs to move away from the reservoir and depend continuously on the reservoir profile which is guaranteed by Theorem 4.1. Away from the reservoir, the BIF can be linearized around the flat solutions by similar arguments used in analyzing the BIF operators near $s = s_0 \equiv 0$. Again the explicit knowledge of relevant symbols will be an integral part of the process. In a future paper, we use the results and ideas derived here to show the principle of linearized stability can be applied to the BIF of the two-dimensional Case II diffusion model.

8. **Appendix. Proof of Theorem 5.7**

*Proof.* Let $v, w, s \in H_p^2([0, 1])$ and define the integral kernel, $K_s$, as follows

$$K_s(x, \tilde{x}) = \frac{1}{2\pi} \log \left| \frac{2 \cosh\left(2\pi\left(s(x) - s(\tilde{x})\right)\right) - 2 \cos\left(2\pi(x - \tilde{x})\right)}{2 - 2 \cos\left(2\pi(x - \tilde{x})\right)} \right|$$

Using the reflection relationship between $G(x, \tilde{x})$ and $G(x, -\tilde{x})$ one has

$$G\left(x - \tilde{x}, s(x) - s(\tilde{x})\right) = K_s(x, \tilde{x}) + G(x - \tilde{x}, 0) + s(\tilde{x}) - s(x)$$

$$\left(Op_{v(x)-v(\tilde{x})}(a_G)[f] - Op_{w(x)-w(\tilde{x})}(a_G)[f]\right)(x)$$

$$= \int_0^1 \left[w(x) - w(\tilde{x}) - \left(v(x) - v(\tilde{x})\right)\right] f(\tilde{x}) d\tilde{x}$$

$$+ \int_0^1 \left[K_v(x, \tilde{x}) + G(x - \tilde{x}, 0)\right] f(\tilde{x}) d\tilde{x}$$

$$- \int_0^1 \left[K_w(x, \tilde{x}) + G(x - \tilde{x}, 0)\right] f(\tilde{x}) d\tilde{x} \quad (51)$$

Since, $K_s$ is an even function with respect to the argument $s(x) - s(\tilde{x})$ adding and subtracting the terms, $|v(x) - v(\tilde{x})|$ and $|w(x) - w(\tilde{x})|$ to the first and second integral kernels in (51) respectively yields

$$\left(Op_{v(x)-v(\tilde{x})}(a_G)[f] - Op_{w(x)-w(\tilde{x})}(a_G)[f]\right)(x)$$
$$= \int_0^1 \left[w(x) - w(\tilde{x}) - \left(v(x) - v(\tilde{x})\right)\right]f(\tilde{x})d\tilde{x}$$
$$+ \int_0^1 \left[|v(x) - v(\tilde{x})| - |w(x) - w(\tilde{x})|\right]f(\tilde{x})d\tilde{x}$$
$$+ Op_{|v(x)-v(\tilde{x})|}[f](x) - Op_{|w(x)-w(\tilde{x})|}(a_G)[f](x) \quad (52)$$

Using Sobolev embedding theorem there exists a nonnegative constant, $C$, such that

$$\left\|\left(Op_{v(\cdot)-v(\tilde{x})}(a_G)[f] - Op_{w(\cdot)-w(\tilde{x})}(a_G)[f]\right)(\cdot)\right\|_{L_{2,p}}$$
$$\leq 2C\|f\|_{L_{2,p}}\|v - w\|_{H_p^2} + \left\|\left(Op_{|v(\cdot)-v(\tilde{x})|}[f] - Op_{|w(\cdot)-w(\tilde{x})|}(a_G)[f]\right)(\cdot)\right\|_{L_{2,p}}$$

Hence all we need is a Lipschitz-type estimate for

$$Op_{|v(\cdot)-v(\tilde{x})|}[f] - Op_{|w(\cdot)-w(\tilde{x})|}(a_G)[f]$$

Using Definition 5.2

$$Op_{|v(x)-v(\tilde{x})|}[f](x) - Op_{|w(x)-w(\tilde{x})|}(a_G)[f](x) =$$
$$Op\left(e^{-2\pi|k||w(x)-w(\tilde{x})|}F_r(x,\tilde{x},k)\right)[-r(x,\tilde{x})f(\tilde{x})]$$

where $r(x,\tilde{x}) = |v(x) - v(\tilde{x})| - |w(x) - w(\tilde{x})|$ and

$$F_r(x,\tilde{x},k) = \sum_{j=0}^{\infty} \frac{\left(-2\pi|k|r(x,\tilde{x})\right)^j}{(j+1)!}$$

To complete the proof we show

$$e^{-2\pi|k||w(x)-w(\tilde{x})|}F_r(x,\tilde{x},k) \in S^0_{1,\frac{1}{2},\frac{1}{2}}$$

Assume without loss of generality $r = r(x,\tilde{x}) \geq 0$, $k \in \mathbb{Z}^+$ and define

$$\phi(z) = \begin{cases} 1, & z = 0 \\ \frac{1-\exp(-2\pi z)}{2\pi z}, & z > 0 \end{cases}$$

Then one can compute

$$\triangle_k\left(F_r(x,\tilde{x},k)\right) = \frac{e^{-2\pi rk}}{k+1}\phi(r) - \frac{1}{k+1}\phi(rk)$$
$$\triangle_k^2\left(F_r(x,\tilde{x},k)\right) = \frac{e^{-2\pi rk}\left(k(e^{-2\pi r}-1) + e^{-2\pi r} - 3\right)\phi(r) + 2\phi(rk)}{(k+1)(k+2)}$$

Using triangle inequality along with the fact that $\phi(z)$ is bounded on $[0,\infty)$ yields a constant $C$ independent of $k$ such that

$$\sup_{r\geq0}\left|\triangle_k\left(F_r(x,\tilde{x},k)\right)\right|\langle k\rangle \leq C$$
$$\sup_{r\geq0}\left|\triangle_k^2\left(F_r(x,\tilde{x},k)\right)\right|\langle k\rangle^2 \leq C$$

Using Lemma 5.4 one gets the continuity estimates needed for $e^{-2\pi|k||w(x)-w(\tilde{x})|}$ in Definition 5.1. Furthermore, $\triangle_k^\alpha F_r(x,\tilde{x},k)$ for $\alpha = 0, 1, 2$ is comprised of several

terms with a similar structure to $e^{-2\pi|k|z}$ and $1 - e^{-2\pi|k|z}$ and it possesses the necessary decay properties in $k$ to satisfy Definition 5.1. Again using Lemma 5.4 yields

$$e^{-2\pi|k||w(x)-w(\tilde{x})|}F_r(x,\tilde{x},k) \in S^0_{1,\frac{1}{2},\frac{1}{2}}$$

□

**Proof of Theorem 5.8**

*Proof.* Let $f \in L_{2,p}([0,1])$ and $h \in H^1_p([0,1])$. In (27) let $J = G$ then

$$\left\|\left[Op_{s_n(\cdot)}(a_G) - Op_{s(\cdot)}(a_G)\right][f]\right\|_{L_{2,p}} =$$

$$= \left\|\int_0^1 \sum_{k\in\mathbb{Z}^*} \frac{e^{-2\pi|k|s_n(\cdot)} - e^{-2\pi|k|s(\cdot)}}{2\pi|k|} e^{2\pi ik(\cdot-\tilde{x})}f(\tilde{x})d\tilde{x}\right\|_{L_{2,p}}$$

$$\leq \sup_{y\in[0,1]} \left\|\frac{e^{-2\pi|\cdot|s_n(y)} - e^{-2\pi|\cdot|s(y)}}{2\pi|\cdot|}\right\|_{l^*_\infty}\|f\|_{L_{2,p}}$$

$$= \sup_{k\in\mathbb{Z}^*,y\in[0,1]} \left|\int_{s(y)}^{s_n(y)} \frac{d}{d\sigma}\frac{e^{-2\pi|k|\sigma}}{2\pi|k|}d\sigma\right|\|f\|_{L_{2,p}}$$

$$\leq \sup_{y\in[0,1]} |s_n(y) - s(y)|\|f\|_{L_{2,p}} \leq \|s_n - s\|_{H^2_p}\|f\|_{L_{2,p}}$$

In (27) let $J = K$, then

$$\left\|\left[Op_{s_n(\cdot)}(a_K) - Op_{s(\cdot)}(a_K)\right][f]\right\|_{L_{2,p}}$$

$$= \sup_{k\in\mathbb{Z}^*,y\in[0,1]} \left|\int_{s(y)}^{s_n(y)} \frac{d}{d\sigma}e^{-2\pi|k|\sigma}d\sigma\right|\|f\|_{L_{2,p}} \leq C\|s_n - s\|_{H^2_p}\|f\|_{L_{2,p}}$$

To prove (28), first define the operator

$$Op\left(e^{-2\pi|k|s(\tilde{x})}\frac{e^{-2\pi|k|\left(|s_n(\tilde{x})|-|s(\tilde{x})|\right)} - 1}{2\pi|k|}\right)[f] \tag{53}$$

and we assume without loss of generality $(|s_n(\tilde{x})| - |s(\tilde{x})|) \geq 0$. We considered (53) in Theorem 5.7 where we showed that

$$\left\|Op\left(e^{-2\pi|k|s(\tilde{x})}\frac{e^{-2\pi|k|\left(|s_n(\tilde{x})|-|s(\tilde{x})|\right)} - 1}{2\pi|k|}\right)[f]\right\|_{L_{2,p}} \leq C\|s_n - s\|_{H^2_p}\|f\|_{L_{2,p}}$$

This inequality combined with a composition argument gives

$$\left\|Op\left(e^{-2\pi|k|s(\tilde{x})}\left[e^{-2\pi|k|(|s_n(\tilde{x})|-|s(\tilde{x})|)} - 1\right]\right)[h]\right\|_{L_{2,p}} \leq C\|s_n - s\|_{H^2_p}\|h\|_{H^1_p}$$

A few computations produce

$$\left(Op_{-s_n(\tilde{x})}(a_G) - Op_{-s(\tilde{x})}(a_G)\right)[f]$$

$$= Op\left(e^{-2\pi|k|s(\tilde{x})}\frac{e^{-2\pi|k|\left(|s_n(\tilde{x})|-|s(\tilde{x})|\right)} - 1}{2\pi|k|}\right)[f]$$

$$\left(Op_{-s_n(\tilde{x})}(a_K) - Op_{-s(\tilde{x})}(a_K)\right)[h]$$
$$= Op\left(e^{-2\pi|k|s(\tilde{x})}\left(e^{-2\pi|k|\left(|s_n(\tilde{x})|-|s(\tilde{x})|\right)} - 1\right)\right)[h]$$

Thus, we have proved (28) for $J = G, K$. To complete the proof of the theorem, suppose

$$r := |s_n(x) - s_n(\tilde{x})| - |s(x) - s(\tilde{x})| \leq 0$$

then one has

$$Op_{s_n(x)-s_n(\tilde{x})}(a_K)[h] - Op_{s(x)-s(\tilde{x})}(a_K)[h] =$$
$$Op\left((-2\pi|k|)e^{-2\pi|k||s(x)-s(\tilde{x})|}F_k(r)\right)[rh]$$

where we defined $F_k(r)$ in Theorem 5.7. Using the same argument as in (28) we have the norm estimate (29). If $r := |s_n(x) - s_n(\tilde{x})| - (|s(x) - s(\tilde{x})|) > 0$, then one has

$$Op_{s_n(x)-s_n(\tilde{x})}(a_K)[h] - Op_{s(x)-s(\tilde{x})}(a_K)[h] =$$
$$Op\left((-2\pi|k|)e^{-2\pi|k||s_n(x)-s_n(\tilde{x})|}(-F_k(r))\right)[rh]$$

and the proof remains the same. Due to similar arguments, we omitted the proof of the $J, J' = H$ cases in (27), (28) and (29). □

<div align="center">REFERENCES</div>

[1] R. A. Adams, "Sobolev Spaces," Academic Press, 1985.

[2] T. Alfrey, E. F. Gurnee and W. G. Lloyd, *Diffusion in glassy polymers*, Journal of Polymer Science, **12** (1966), 249–261.

[3] S. Angenent, *Constructions with analytic semigroups and abstract exponential decay results for eigenfunctions*, in "Topics in Nonlinear Analysis," Birkhäuser, (1999), 11–27.

[4] G. Astarita and G. C. Sarti, *A class of mathematical models for sorption of swelling solvents in glassy polymers*, Polymer Engineering and Science, **18** (1978), 388–395.

[5] D. S. Cohen and T. Erneux, *Free boundary probelms in controlled release pharmaceuticals. I. Diffusion in glassy polymers*, SIAM Journal of Applied Mathematics, **48** (1988), 1451–1465.

[6] P. Guidotti and J. Palesko, *Transient instability in case II diffusion*, Journal of Polymer Science, Part B: Polymer Physics, **36** (1998), 2941–2947.

[7] P. Guidotti, *Diffusion in glassy polymers: Sorption of a finite amount of swelling solvent*, Nonlinear Differential Equations and Applications, **6** (1999), 297–317.

[8] P. Guidotti, *2-D free boundary value problem with onset of a phase and singular elliptic boundary value problem*, Journal of Evolution Equations., **2** (2002), 395–424.

[9] P. Guidotti, *A new first kind boundary integral formulation for the dirichlet-to-neumann map in 2D*, Journal of Computational Physics, **190** (2003), 325–345.

[10] P. Guidotti, *A class of free boundary problems with onset of a new phase*, SIAM Journal on Mathematical Analysis, **38** (2007), 1981–2004.

[11] H. Kumano-Go and M. Nagase, *Pseudo-differential operators with non-regular symbols and applications*, Funkcialaj Ekvacioj, **21** (1978), 151–192.

[12] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser Verlag, 1995.

[13] M. Nagase, *The $L^p$-boundedness of pseudo-differential operators with non-regular symbols*, Communications in Partial Differential Equations, **2** (1977), 1045–1061.

[14] J.Saranen and G. Vainikko, "Periodic Integral and Pseudodifferential Equations with Numerical Approximation," Springer, 2002

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