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Wellposedness of a nonlocal nonlinear diffusion equation of image processing

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ABSTRACT

Existence and uniqueness of solutions to non-smooth initial data are established for a slight modification of the degenerate regularization of the well-known Perona–Malik equation first proposed in Guidotti and Lambers (2009). The results heavily rely on the choice of an appropriate functional setting inspired by a recent approach to degenerate parabolic equations via so-called singular Riemannian manifolds (Amann, 2013, 2016).

1. Introduction

In the early 90s, P. Perona and J. Malik [18] introduced a novel paradigm by proposing the use of nonlinear diffusion as an image processing tool. The stark contrast between the numerical effectiveness of their method and its mathematical ill-posedness, see [16], spurred significant subsequent research in mathematics and image processing. A number of mathematical “fixes” have been proposed over the past decades. It is referred to [12] for an overview. Of relevance for this article is the fractional derivatives’ based regularization proposed in [14]. While it is well-posed as a quasi-linear parabolic equation, it appears so only

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in a smooth context (i.e. for smooth enough initial data). Characteristic functions or linear combinations thereof are, however, of extreme interest in applications since edges (sharp transitions in gray level) are an essential feature of many images. Mathematical results in function spaces which contain them are therefore desirable. As, even in the corresponding linear case, uniqueness may fail to hold (see [13] for an illustration), the careful choice of functional setting is paramount. It has indeed been impossible thus far to identify an appropriate concept of weak solution yielding well-posedness for a class of initial data large enough to include characteristic functions of smooth sets. Allowing for non-smooth initial data readily leads to degenerate parabolic equations. The precise degeneration type, however, depends on the exact properties of the chosen non-smooth initial datum. The construction of a unique solution proposed here is therefore based on the use of function spaces defined around a fixed singular function (in order to fix the degeneration type) and of recently developed results for parabolic equations on singular Riemannian manifolds which provide a tool for analyzing degenerate parabolic equations with fixed degeneration; see [2,4,21,22]. While the results of this paper do not resolve the general uniqueness/non-uniqueness question, they appear to be the first delivering non-trivial existence results of solution to non-smooth initial data and uniqueness in a restricted class of functions which share a common singularity.

The remainder of the paper is organized as follows: Results about maximal regularity for parabolic equations and weighted function spaces are presented in Section 2. Local well-posedness of the nonlinear model is shown in Section 3 and global well-posedness is established in Section 4 by means of the principle of linearized stability for small perturbations of the non-smooth initial datum. The main results are formulated in Theorems 3.14, 3.19, and 4.6.

1.1. Notations

For $s \geq 0$ and $p \in (1, \infty)$, we denote by $\mathfrak{F}^s(\mathbb{R}^N)$ the function spaces obtained by replacing $\mathfrak{F}$ by W_p or BC. If Q^N is the N-dimensional unit cube, the spaces $\mathfrak{F}^s_\pi(\mathsf{Q}^N)$ are the corresponding subspaces consisting of periodic functions with periodicity box given by Q^N.

Given any topological set U, $\mathring{U}$ denotes the interior of U.

For any two Banach spaces X, Y, $X \doteq Y$ means that they are equal in the sense of equivalent norms. The notation $\mathcal{L}\mathrm{is}(X, Y)$ stands for the set of all bounded linear isomorphisms from X to Y.

The symbol $\sim$ always denotes Lipschitz equivalence. We write $\dot{\mathbb{N}} = \mathbb{N} \setminus \{0\}$.

2. Maximal L_p-regularity in a weighted L_p-framework

2.1. Transforming the problem onto the torus

Let $N = 1, 2$, define $\mathsf{Q}^N = [-1, 1)^N$, and consider the following problem:

$$\begin{cases} \partial_t u - \mathrm{div}\big(\alpha_\varepsilon(u)\nabla u\big) = 0 & \text{in } \mathsf{Q}^N \times (0, \infty), \\ u & \text{periodic}, \\ u(0) = u_0 & \text{in } \mathsf{Q}^N, \end{cases} \tag{2.1}$$

where $\alpha_\varepsilon(u) = \big[1 + |\nabla^{1-\varepsilon} u|^2\big]^{-1}$ and $\varepsilon \in (0, 1)$. A precise definition of the fractional derivative appearing in the nonlinear coefficient α_ε will be given in Section 3.

We shall be interested in non-smooth initial data u_0 for which $\alpha_\varepsilon(u_0)$ vanishes on a C^3-submanifold $\Gamma \subset \mathring{\mathsf{Q}}^N$ of codimension 1 (which may not be connected). For δ sufficiently small, we can always choose a 2δ-tubular neighborhood $\mathscr{U}_{2\delta} \subset\subset \mathring{\mathsf{Q}}^N$ of any such Γ, even if Γ has merely C^2 boundary. Define

$\mathsf{d} \in C^3(\mathsf{Q}^N \setminus \Gamma, (0,1])$ by

$$\mathsf{d}(x) = \begin{cases} \mathsf{dist}(x,\Gamma), & \text{in } \mathscr{U}_\delta \setminus \Gamma, \\ 1, & \text{in } \mathsf{Q}^N \setminus \mathscr{U}_{2\delta}, \end{cases} \tag{2.2}$$

and observe that $\mathsf{dist}(x,\Gamma)$ is well-defined and C^3 for δ sufficiently small.

Considering $x_1, x_2 \in \mathbb{R}^N$ to be equivalent if $x_1 - x_2 = 2m$ for some $m \in \mathbb{Z}^N$, let ϕ be the projection mapping taking $x \in \mathbb{R}^N$ to its equivalence class. It clearly holds that $\phi(\mathsf{Q}^N) = \mathbb{T}^N$, where $\mathbb{T}^N$ is the N-dimensional torus.

Throughout the rest of this paper, unless stated otherwise, we always assume that

- $s \geq 0$, $k \in \dot{\mathbb{N}}$, $1 < p \leq \infty$ and $\vartheta \in \mathbb{R}$.
- $\mathfrak{F} = W_p$ for $1 < p < \infty$, or $\mathfrak{F} = BC$.

Remark 2.1. If we equip $\mathbb{T}^N$ with the metric $\phi_* g_N$, where g_N is the N-dimensional Euclidean metric on Q^N and $\phi_* = \left[(\phi|_{\mathsf{Q}^N})^{-1}\right]^*$, i.e. $\phi_* g_N$ is the pullback metric along $(\phi|_{\mathsf{Q}^N})^{-1}$, then $(\mathbb{T}^N, \phi_* g_N)$ is a closed smooth manifold. Therefore, any periodic function space $\mathfrak{F}^s_\pi(\mathsf{Q}^N)$ defined on (Q^N, g_N) is isomorphic to the corresponding $\mathfrak{F}^s(\mathbb{T}^N)$ defined on $(\mathbb{T}^N, \phi_* g_N)$. So all well-established function space theory results, such as those pertaining to interpolation and to lifting properties, transfer to the spaces $\mathfrak{F}^s_\pi(\mathsf{Q}^N)$. See, for instance, [25, Chapter 7] for more details on function space theory on closed manifolds.

We let $\Gamma_{\mathbb{T}} = \phi(\Gamma)$ and set

$$(\mathsf{M}, g) = (\mathbb{T}^N \setminus \Gamma_{\mathbb{T}}, \phi_* g_N|_{\mathbb{T}^N \setminus \Gamma_{\mathbb{T}}}).$$

Denote the metrics g_N and $\phi_* g_N$ by $(\cdot|\cdot)$ and $(\cdot|\cdot)_g$, and the norms induced by g_N and $\phi_* g_N$ by $|\cdot|$ and $|\cdot|_g$, respectively.

As long as it causes no confusion, we will denote the usual covariant derivative, divergence, and Laplacian on (Q^N, g_N) as well as their restrictions to $(\mathsf{Q}^N \setminus \Gamma, g_N)$ by ∇, div and Δ respectively. Similarly, ∇_g, div_g and Δ_g will denote their counterparts on both $(\mathbb{T}^N, \phi_* g_N)$ and (M, g).

Now problem (2.1) can be equivalently stated as

$$\begin{cases} \partial_t u - \mathrm{div}_g(\alpha_\varepsilon(u)\nabla_g u) = 0 & \text{in } \mathbb{T}^N \times (0,\infty), \\ u(0) = u_0 & \text{in } \mathbb{T}^N. \end{cases} \tag{2.3}$$

Here it is understood that $\alpha_\varepsilon(u) = \phi_* \alpha_\varepsilon(\phi^* u)$.

2.2. Periodic weighted function spaces

Note that the function defined by

$$\rho(x) = \mathsf{d}(y), \quad y \in \phi^{-1}(x) \cap \mathsf{Q}^N, \tag{2.4}$$

is well-defined on M and satisfies $\rho \in C^3(\mathsf{M}, (0,1])$. We will begin with the definition of weighted function spaces on (M, g) (see [2,3]) in order to derive the definition of the corresponding weighted periodic function spaces on $\mathsf{Q}^N \setminus \Gamma$.

Given an arbitrary finite dimensional Hilbert space X, denote its inner product by $(\cdot|\cdot)_X$. The weighted Sobolev space of X-valued functions $W^{k,\vartheta}_p(\mathsf{M}, X)$ is defined as the completion of $\mathcal{D}(\mathsf{M}, X)$, the space of X-valued test-functions, with respect to the norm

$$\|\cdot\|_{k,p;\vartheta} : u \mapsto \left(\sum_{i=0}^k \|\rho^{\vartheta+i} |\nabla^i_g u|_g\|_p^p \right)^{\frac{1}{p}},$$

with the understanding that $L_p^\vartheta(\mathsf{M},X) = W_p^{0,\vartheta}(\mathsf{M},X)$ and that $\nabla_g^{i+1}u := \nabla_g \circ \nabla_g^i u$. The weighted Sobolev–Slobodeckii spaces are defined as

$$W_p^{s,\vartheta}(\mathsf{M},X) := \Big(L_p^\vartheta(\mathsf{M},X), W_p^{k,\vartheta}(\mathsf{M},X)\Big)_{s/k,p},$$

for $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $k = [s]+1$. Here $(\cdot,\cdot)_{\theta,p}$ is the standard real interpolation method [1, Chapter I.2.4.1]. Define

$$BC^{k,\vartheta}(\mathsf{M},X) := \Big(\{u \in C^k(\mathsf{M},X) : \|u\|_{k,\infty;\vartheta} < \infty\}, \|\cdot\|_{k,\infty;\vartheta}\Big), \tag{2.5}$$

where $\|u\|_{k,\infty;\vartheta} := \max_{0\le i\le k} \|\rho^{\vartheta+i}|\nabla_g^i u|_g\|_\infty$.

Remark 2.2. Note that (M,g) is an incomplete manifold. Indeed, by [5, Lemma 3.4] and [20, Proposition 12], $(\mathsf{M},g;\rho)$ can be seen as a C^2-singular manifold. It follows that the weighted function spaces introduced above are all well-defined for (M,g) (cf. [2,3,21]). The properties of weighted function spaces defined on C^2-singular manifolds established in the cited references are all inherited by the weighted function spaces $\mathfrak{F}^{s,\vartheta}(\mathsf{M},X)$.

We can define periodic weighted function spaces on $(\mathsf{Q}^N \setminus \Gamma, g_N)$ in the same manner just by replacing the weight function ρ, ∇_g and $|\cdot|_g$ by d, ∇ and $|\cdot|$, respectively. We denote these spaces by $\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q}^N \setminus \Gamma, X)$. By the identification

$$\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q}^N \setminus \Gamma, X) \doteq \phi^*\mathfrak{F}^{s,\vartheta}(\mathsf{M},X), \tag{2.6}$$

the space $\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q}^N \setminus \Gamma, X)$ enjoys the same properties as $\mathfrak{F}^{s,\vartheta}(\mathsf{M},X)$. For notational brevity, we still denote the norms of the weight function spaces $\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q}^N \setminus \Gamma, X)$ by $\|\cdot\|_{k,p;\vartheta}$ and $\|\cdot\|_{k,\infty;\vartheta}$, respectively.

Lemma 2.3. *Let $s \ge 0$ for $\mathfrak{F} = W_p$ or $s \in \mathbb{N}$ for $\mathfrak{F} = BC$ and $\vartheta \in \mathbb{R}$, then it holds that*

(i) $\nabla \in \mathcal{L}\Big(\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q}^N \setminus \Gamma, \mathbb{R}), \mathfrak{F}_\pi^{s-1,\vartheta+1}(\mathsf{Q}^N \setminus \Gamma, \mathbb{R}^N)\Big)$.

(ii) $\operatorname{div} \in \mathcal{L}\Big(\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q}^N \setminus \Gamma, \mathbb{R}^N), \mathfrak{F}_\pi^{s-1,\vartheta+1}(\mathsf{Q}^N \setminus \Gamma, \mathbb{R})\Big)$.

Proof. (i) follows from [2, Theorem 7.5], (2.5) and (2.6). (ii) is a consequence of [21, Propositions 2.5, 2.8].

Notice that there is a difference in weights between the above lemma and the Refs. [2, Theorem 7.5] and [21, Proposition 2.8]. The former is due to the fact that weights in [2] depend on whether sections are defined on the tangent or cotangent bundle and we do not distinguish between them here. The latter stems from the fact the divergence operator is defined for sections of the tangent bundle, while the covariant derivative naturally maps into the cotangent bundle and going between the two requires a change of weight (see [21, Proposition 2.5]). □

Lemma 2.4. *For $\vartheta' \in \mathbb{R}$ and s, ϑ as in Lemma 2.3, we have that*

$$[u \mapsto \rho^\vartheta u] \in \mathcal{L}\mathrm{is}\Big(\mathfrak{F}_\pi^{s,\vartheta'+\vartheta}(\mathsf{Q}^N \setminus \Gamma, X), \mathfrak{F}_\pi^{s,\vartheta'}(\mathsf{Q}^N \setminus \Gamma, X)\Big).$$

Proof. See [21, Propositions 2.4] and (2.6). □

Lemma 2.5. *Let $s \le k \in \dot{\mathbb{N}}$ and $\vartheta_i \in \mathbb{R}$ with $i = 0,1$. $[(u,v) \mapsto (u|v)_X]$ is a continuous bilinear map in each of the following functional settings*

$$W_{p,\pi}^{s,\vartheta_0}(\mathsf{Q}^N \setminus \Gamma, X) \times BC_\pi^{k,\vartheta_1}(\mathsf{Q}^N \setminus \Gamma, X) \to W_{p,\pi}^{s,\vartheta_0+\vartheta_1}(\mathsf{Q}^N \setminus \Gamma) \quad \textit{or}$$
$$BC_\pi^{k,\vartheta_0}(\mathsf{Q}^N \setminus \Gamma, X) \times BC_\pi^{k,\vartheta_1}(\mathsf{Q}^N \setminus \Gamma, X) \to BC_\pi^{k,\vartheta_0+\vartheta_1}(\mathsf{Q}^N \setminus \Gamma).$$

Proof. This follows from [3, Theorem 13.5] and (2.6). □

Lemma 2.6. *Suppose that $k_i \in \mathbb{N}$, $\vartheta_i \in \mathbb{R}$ with $i = 0, 1$, $0 < \theta < 1$ and $k_0 < k_1$. Then*

$$\left(W^{k_0,\vartheta_0}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma, X), W^{k_1,\vartheta_1}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma, X)\right)_{\theta,p} \doteq W^{k_\theta,\vartheta_\theta}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma, X),$$

where $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$ for any $\xi_0, \xi_1 \in \mathbb{R}$ and the case $k_\theta \in \mathbb{N}$ needs to be excluded.

Proof. It follows from [3, Theorem 8.2(i), formulas (8.3), (21.2)] and (2.6). □

Proposition 2.7. *Suppose that $s > k + \frac{N}{p}$ and $\vartheta \in \mathbb{R}$. Then*

$$W^{s,\vartheta}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma, X) \hookrightarrow BC^{k,\vartheta+\frac{N}{p}}_{\pi}(\mathsf{Q}^N \setminus \Gamma, X).$$

Proof. See [2, Theorem 14.2(ii)] and (2.6). □

2.3. Maximal regularity of type L_p

In this subsection, we will state some preliminary concepts and results of maximal L_p-regularity for differential operators and their application to quasi-linear parabolic equations. The reader is referred to [1, 6], and [9] for more details about these concepts.

We consider the following abstract Cauchy problem

$$\begin{cases} \partial_t u(t) + \mathcal{A}u(t) = f(t), \; t \geq 0 \\ u(0) = 0. \end{cases} \tag{2.7}$$

For $\theta \in (0, \pi]$, the open sector of angle 2θ is denoted by

$$\Sigma_\theta := \{\omega \in \mathbb{C} \setminus \{0\} : |\arg \omega| < \theta\}.$$

Definition 2.8. Let X be a complex Banach space, and $\mathcal{A}$ be a densely defined closed linear operator in X with dense range. $\mathcal{A}$ is called sectorial if $\Sigma_\theta \subset \rho(-\mathcal{A})$ for some $\theta > 0$ and

$$\sup\{|\mu(\mu + \mathcal{A})^{-1}| : \mu \in \Sigma_\theta\} < \infty.$$

The class of sectorial operators in X is denoted by $\mathcal{S}(X)$. The spectral angle $\phi_{\mathcal{A}}$ of $\mathcal{A}$ is defined by

$$\phi_{\mathcal{A}} := \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-\mathcal{A}), \sup_{\mu \in \Sigma_{\pi-\phi}} |\mu(\mu + \mathcal{A})^{-1}| < \infty.\}.$$

Definition 2.9. Assume that $X_1 \overset{d}{\hookrightarrow} X_0$ is some densely embedded pair of Banach spaces. Suppose that $\mathcal{A} \in \mathcal{S}(X_0)$ with $dom(\mathcal{A}) = X_1$. Then, the Cauchy problem (2.7) is said to have maximal L_p-regularity if, for any

$$f \in L_p(\mathbb{R}_+, X_0),$$

Eq. (2.7) has a unique solution

$$u \in L_p(\mathbb{R}_+, X_1) \cap H^1_p(\mathbb{R}_+, X_0).$$

We denote this by

$$\mathcal{A} \in \mathcal{MR}_p(X_1, X_0).$$

Maximal regularity theory has proven a powerful tool in the theory of nonlinear parabolic equations. We will apply it to the study of problem (2.1). To this end, let us consider the following abstract evolution equation

$$\begin{cases} \partial_t u + \mathcal{A}(u)u = f(u), & t \geq 0, \\ u(0) = 0, \end{cases} \tag{2.8}$$

in X_0. We have the following existence and uniqueness result for Eq. (2.8).

Theorem 2.10 ([7, *Theorem 2.1*]). *Let $1 < p < \infty$ and $X_1 \overset{d}{\hookrightarrow} X_0$ be a densely embedded pair of Banach spaces. Setting $X_{1/p} := (X_0, X_1)_{1-1/p,p}$, suppose that $U \subset X_{1/p}$ is open and that $\mathcal{A}, f$ satisfy*

$$(\mathcal{A}, f) \in C^{1-}\big(U, \mathcal{MR}_p(X_1, X_0) \times X_0\big).$$

Then for every $u_0 \in U$, there exist $T = T(u_0) > 0$ and a unique solution of (2.8) on $J = [0, T]$ with

$$u \in L_p(J, X_1) \cap H^1_p(J, X_0).$$

3. Local well-posedness of the nonlinear model

In this section, we will establish the existence and uniqueness of solutions to (2.1). The precise definition of the fractional gradient $\nabla^{1-\varepsilon}$ in the one and two dimensional cases will be stated separately in the following two subsections. In order to allow for non-smooth initial data for (2.1) and the corresponding degeneration in the diffusion coefficient they cause, it is necessary to resort to weighted spaces. We put

$$E_0 := L^{\vartheta+2\varepsilon}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma), \qquad E_1 := W^{2,\vartheta}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma),$$

and

$$E_{\frac{1}{p}} := (E_0, E_1)_{1-1/p,p} = W^{2-2/p,\vartheta+\frac{2\varepsilon}{p}}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma).$$

Throughout the rest of this section, we always assume

$$\vartheta \leq -2,\ p > \max\left\{\frac{N+2}{\varepsilon}, -\frac{N+2\varepsilon}{\vartheta}\right\}, \quad \varepsilon \neq \frac{1}{2} \tag{3.1}$$

or

$$\vartheta = -2\varepsilon,\ p > \max\left\{\frac{2N+2}{\varepsilon}, \frac{4N+5}{2}\right\}, \quad \varepsilon > 1 - \frac{1}{2p}. \tag{3.2}$$

Conditions (3.1) and (3.2) are imposed in order for technically necessary embeddings to be valid. Notice that the first condition allows for more freedom in the choice of ε, whereas the second will make it possible to obtain stronger results (see Section 4).

If (3.1) holds, then it is not hard to verify by the definition of $W^{k,\vartheta}_p(\mathsf{Q}^N \setminus \Gamma)$ in Section 2.2 and the choice of ϑ that

$$E_0 \hookrightarrow L_{p,\pi}(\mathsf{Q}^N) \quad \text{and} \quad E_1 \hookrightarrow W^2_{p,\pi}(\mathsf{Q}^N).$$

Interpolation theory implies that

$$E_{\frac{1}{p}} \hookrightarrow W^{2-2/p}_{p,\pi}(\mathsf{Q}^N). \tag{3.3}$$

We define $\mathbb{R}_\Gamma$ to be the set of functions which are a constant on each component of $\mathsf{Q}^N \setminus \Gamma$.

Lemma 3.1. *Assume that (3.2) is satisfied. Then*

$$E_1 \hookrightarrow W^2_{p/2,\pi}(\mathsf{Q}^N).$$

Proof. First note that for p satisfying (3.2), by Proposition 2.7, one has that

$$E_1 \hookrightarrow BC^{1,\vartheta+\frac{N}{p}}_{\pi}(\mathsf{Q}^N \setminus \Gamma).$$

Therefore, any $u \in E_1$ admits a smooth trace $\gamma_\Gamma(u)$ and $\gamma_\Gamma(u) = 0$; similarly $\gamma_\Gamma(|\nabla u|) = 0$, in view of the assumptions on the parameter ϑ. These estimates imply that, on each component of $\mathsf{Q}^N \setminus \Gamma$, u and ∇u admit unique continuous extensions onto Γ and thus that $E_1 \hookrightarrow BC^1_\pi(\mathsf{Q}^N)$.

Pick $q < p$. First, it is clear that $E_1 \hookrightarrow L_{q,\pi}(\mathsf{Q}^N)$ since, by definition,

$$\begin{aligned}\int_{\mathsf{Q}^N} |\nabla^2 u|^q\, dx &\le \int_{\mathsf{Q}^N} |\mathsf{d}^{2\varepsilon-2}\mathsf{d}^{2-2\varepsilon}\nabla^2 u|^q\, dx \\ &\le \Big[\int_{\mathsf{Q}^N} |\mathsf{d}^{2\varepsilon-2}|^{\frac{qp}{p-q}}\, dx\Big]^{\frac{p-q}{p}} \Big[\int_{\mathsf{Q}^N} |\mathsf{d}^{2-2\varepsilon}\nabla^2 u|^p\, dx\Big]^{q/p}.\end{aligned}$$

To make the first term on the right hand side of the inequality finite, it suffices to require that $(2\varepsilon-2)\frac{qp}{p-q} > -1$. This is clear for $N = 1$ where the singularity is at isolated points, whereas for $N = 2$ it follows from the fact that the singularity is along a smooth curve. The above parameter inequality is equivalent to

$$\varepsilon > 1 - \frac{1}{2q} + \frac{1}{2p}.$$

Taking $q = p/2$ yields $\varepsilon > 1 - \frac{1}{2p}$. $\square$

The assumption $p > \frac{4N+5}{2}$ in (3.2) guarantees that $p > \frac{2N+2}{\varepsilon}$ and $\varepsilon > 1 - \frac{1}{2p}$ do not conflict.

3.1. One dimensional case

Since we are working with periodic functions on $\mathsf{Q} = [-1, 1)$, we can define $\nabla^{1-\varepsilon} = \partial^{1-\varepsilon}$ by means of Fourier series

$$\partial^{1-\varepsilon} u := |\partial|^{-\varepsilon} u' := \mathcal{F}^{-1}\mathrm{diag}\Big\{\frac{i\pi k}{|k|^\varepsilon}\Big\}\mathcal{F}u,$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathrm{diag}\{\frac{i\pi k}{|k|^\varepsilon}\}$ denotes the multiplication operator (in Fourier space) by the function $[k \mapsto \frac{i\pi k}{|k|^\varepsilon}]$.

Lemma 3.2.

$$\partial^{1-\varepsilon} u(x) = \int_{\mathsf{Q}} c_\varepsilon \frac{u'(y)}{|x-y|^{1-\varepsilon}}\, dy + \int_{\mathsf{Q}} h_\varepsilon(x-y) u'(y)\, dy,$$

for some constant $c_\varepsilon > 0$ and $h_\varepsilon \in C^\infty$.

Proof. By definition, one has that

$$\partial^{1-\epsilon} u(x) = \int_{\mathsf{Q}} G_\epsilon(x-y) u'(y)\, dy,$$

with

$$\widehat{G}_\epsilon(k) = \frac{1}{|k|^\epsilon}, \quad k \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}.$$

This means that

$$G_\epsilon(x) = \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^\epsilon} e^{\pi i k x} = 2^{-\varepsilon} \sum_{k\in\mathbb{Z}} \frac{\eta(k/2)}{|k/2|^\epsilon} e^{2\pi i (k/2) x},$$

where $\eta \in C^\infty(\mathbb{R})$ is a cut-off function with

$$\eta(x) = \begin{cases} 0, & |x| \le 1/8, \\ 1, & |x| \ge 1/4. \end{cases}$$

Notice that the Poisson summation formula [24, p. 362] yields, at least formally at first, that

$$\begin{aligned} G_\epsilon(x) &= \sum_{k\in\mathbb{Z}} \frac{\eta(k/2)}{|k|^\epsilon} e^{\pi i k x} \\ &= c|x|^{\varepsilon-1} + c\mathcal{F}\big([\eta(\cdot/2)-1]|\cdot/2|^{-\varepsilon}\big)(x) + c\sum_{k\in\mathbb{Z}^*} \mathcal{F}\big(\eta(\cdot/2)|\cdot/2|^{-\varepsilon}\big)(x+2k), \end{aligned} \tag{3.4}$$

for $x \in [-1,1)$. The second term after the last equality sign is C^∞ as the Fourier transform of a compactly supported function. The function $\mathcal{F}\big(\eta(\cdot/2)|\cdot/2|^{-\epsilon}\big)$ appearing in the last series is rapidly decreasing at infinity (faster than the reciprocal of any polynomial) as the Fourier transform of a smooth function. This rapid decay implies that the last series converges uniformly for x in $[-1,1)$ since none of the addends in the series has a singularity in this interval. It follows in particular that the distribution after the second equality sign in (3.4) is in fact regular, that is, a locally integrable function. Now, while the assumptions of the Poisson summation formula are not met for a straightforward validity of (3.4), it is possible to replace

$$\sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^\epsilon} e^{\pi i k x}$$

by

$$\sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^\epsilon} e^{\pi i k x} e^{-\delta(k/2)^2/2}$$

with $\delta \in (0,1]$ before applying the summation formula to obtain

$$G_\varepsilon^\delta = H_\varepsilon * \frac{1}{\sqrt{2\pi\delta}} e^{-(\cdot)^2/2\delta},$$

where H_ε is the expression after the last equality sign in (3.4). Letting $\delta \to 0$ and using Lebesque's dominated convergence theorem, we obtain the validity of (3.4) itself since G_ε^δ converges to G_ε in the sense of distributions, while the convolution term converges to H_ε in $\mathrm{L}_1(-1,1)$. The claim follows denoting the sum of the last two terms of (3.4) by h_ε. □

Lemma 3.3. *Assume that (3.1) or (3.2) is satisfied. Then*

$$\partial^{1-\varepsilon} \in \mathcal{L}\left(E_{\frac{1}{p}}, BC^1_\pi(\mathsf{Q})\right).$$

Proof. First note that $\mathfrak{F}^s_\pi(\mathsf{Q}) \doteq \phi^*\mathfrak{F}^s(\mathbb{T})$, so function space theory on compact manifolds applies; see [25, Chapter 7].

(i) If (3.1) is assumed to hold, from (3.3), we can infer that

$$\partial^{1-\varepsilon} \in \mathcal{L}\left(E_{\frac{1}{p}}, W^{1+\varepsilon-2/p}_{p,\pi}(\mathsf{Q})\right),$$

since the mapping properties of $\partial^{1-\varepsilon}$ readily follow for the Bessel potential spaces (which can be defined in terms of decay properties of Fourier coefficients) and, then for Sobolev–Slobodeckii spaces as well, by interpolation. Now the statement follows from Sobolev embedding and from $p > \frac{3}{\varepsilon}$ in (3.1).

(ii) If, instead, we assume (3.2), then, Lemma 3.1 and interpolation theory imply that

$$E_{\frac{1}{p}} \hookrightarrow \Big(L_{p/2,\pi}(\mathsf{Q}), W^2_{p/2,\pi}(\mathsf{Q})\Big)_{1-1/p,p} \doteq B^{2-2/p}_{p/2,p,\pi}(\mathsf{Q}).$$

Thus we infer that

$$\partial^{1-\varepsilon} \in \mathcal{L}\left(E_{\frac{1}{p}}, B^{1+\varepsilon-2/p}_{p/2,p,\pi}(\mathsf{Q})\right).$$

Now embedding theorems for Besov spaces and $p > \frac{4}{\varepsilon}$ in (3.2) complete the proof. □

In dimension 1, we are interested in initial data that are close to piecewise constant functions in some proper topology.

It suffices to take the following function H as a generic representative of piecewise constant functions

$$H(x) = \chi_{(-1/2,1/2)}(x) = \begin{cases} 1, & x \in (-1/2, 1/2), \\ 0, & |x| \geq 1/2. \end{cases} \tag{3.5}$$

This means that we choose $\Gamma = \{\pm 1/2\}$ and $H \in \mathbb{R}_\Gamma$.

Proposition 3.4. *The function $\partial^{1-\varepsilon} H$ satisfies*

$$\partial^{1-\varepsilon} H(x) = c_\varepsilon \left[\frac{1}{|x+1/2|^{1-\varepsilon}} - \frac{1}{|x-1/2|^{1-\varepsilon}}\right] + h_\varepsilon(x+1/2) - h_\varepsilon(x-1/2), \quad x \in \mathsf{Q}$$

where $h_\varepsilon \in C^\infty$, for some constant $c_\varepsilon > 0$.

Proof. Using the kernel representation given in Lemma 3.2 and the fact that $H' = \delta_{-1/2} - \delta_{1/2}$ yields that

$$\partial^{1-\varepsilon} H(x) = c_\varepsilon \left[\frac{1}{\left|x+\frac{1}{2}\right|^{1-\varepsilon}} - \frac{1}{\left|x-\frac{1}{2}\right|^{1-\varepsilon}}\right] + h_\varepsilon(x+1/2) - h_\varepsilon(x-1/2), \quad x \in \mathsf{Q},$$

and the claim follows. □

Taking d as in (2.2), there exists some $\mathcal{E} > 1$ such that

$$1/\mathcal{E} < \mathsf{d}^{1-\varepsilon} |\partial^{1-\varepsilon} H| < \mathcal{E}, \quad \text{near } \Gamma. \tag{3.6}$$

We assume that the initial datum is of the form

$$u_0 = H + w_0, \quad w_0 \in E_{\frac{1}{p}}.$$

Remark 3.5. A typical example of the perturbation term w_0 could be $\sin(64\pi x^2)$.

For any $w \in E_{\frac{1}{p}}$, we define

$$\mathscr{A}(w)u := -\mathrm{div}(\alpha_\varepsilon(H+w)\nabla u).$$

Recall that $\alpha_\varepsilon(H+w) := \frac{1}{1+|\partial^{1-\varepsilon}(H+w)|^2}$.

We will apply the theory of $\mathscr{R}$-bounded operators to prove that the operator $\mathscr{A}(w)$ enjoys the property of maximal L_p-regularity.

Definition 3.6. Let X_1 and X_0 be two Banach spaces. A family of operators $\mathcal{T} \in \mathcal{L}(X_1, X_0)$ is called $\mathscr{R}$-bounded, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$ and $x_j \in X_1$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\left|\sum_{j=1}^{N} \varepsilon_j T_j x_j\right|_{L_p(\Omega;X_0)} \leq C \left|\sum_{j=1}^{N} \varepsilon_j x_j\right|_{L_p(\Omega;X_1)}$$

is valid. The smallest such C is called $\mathscr{R}$-bound of $\mathcal{T}$. We denote it by $\mathscr{R}(\mathcal{T})$.

Definition 3.7. Suppose that $\mathcal{A} \in \mathcal{S}(X)$. Then $\mathcal{A}$ is called $\mathscr{R}$-sectorial if there exists some $\phi > 0$ such that

$$\mathscr{R}_{\mathcal{A}}(\phi) := \mathscr{R}\{\mu(\mu + \mathcal{A})^{-1} : \mu \in \Sigma_\phi\} < \infty.$$

The $\mathscr{R}$-angle $\phi^R_{\mathcal{A}}$ is defined by

$$\phi^R_{\mathcal{A}} := \inf\{\theta \in (0,\pi) : \mathscr{R}_{\mathcal{A}}(\pi - \theta) < \infty\}.$$

The class of $\mathscr{R}$-sectorial operators in X is denoted by $\mathcal{RS}(X)$.

Let $R > 0$ and $B_R := \{w \in E_{\frac{1}{p}} : \|w\|_{E_{\frac{1}{p}}} < R\}$.

Lemma 3.8. *There exists a constant C such that*

$$\mathsf{d}^{1-\varepsilon}|\partial^{1-\varepsilon}(H + w)| < C, \quad w \in B_R,$$

and

$$1/C < \mathsf{d}^{2\varepsilon-2}\alpha_\varepsilon(H + w) < C, \quad w \in B_R.$$

Proof. (i) It follows from Lemma 3.3 that

$$|\partial^{1-\varepsilon} w| \quad \text{is uniformly bounded for } w \in B_R,$$

and the boundedness of $\mathsf{d}^{1-\varepsilon}|\partial^{1-\varepsilon}H|$ follows from (3.6). This proves the first assertion.

(ii) We have that

$$\mathsf{d}^{2\varepsilon-2}\alpha_\varepsilon(H + w) = \frac{1}{\mathsf{d}^{2-2\varepsilon} + \mathsf{d}^{2-2\varepsilon}|\partial^{1-\varepsilon}(H + w)|^2}.$$

The first assertion implies the uniform lower bound of the second. It follows from the expression for $\partial^{1-\varepsilon}H$ in Proposition 3.4 that, in a small enough δ-neighborhood $\mathscr{U}_\delta$ of Γ, one has that

$$\mathsf{d}^{1-\varepsilon}|\partial^{1-\varepsilon}H| > \frac{c_\varepsilon}{2},$$

where c_ε is the constant in Proposition 3.4. Clearly, by the uniform boundedness of $|\partial^{1-\varepsilon}w|$ in B_R, it holds that

$$\mathsf{d}^{1-\varepsilon}(x)w(x) \to 0, \quad \text{and} \quad \mathsf{d}^{2-\varepsilon}(x)\frac{d}{dx}w(x) \to 0 \quad \text{as } x \to \Gamma. \tag{3.7}$$

Choosing δ sufficiently small yields $\mathsf{d}^{1-\varepsilon}|\partial^{1-\varepsilon}w| < \frac{c_\varepsilon}{4}$ inside $\mathscr{U}_\delta$. Therefore,

$$\mathsf{d}^{2-2\varepsilon}|\partial^{1-\varepsilon}(H + w)|^2 > \frac{c_\varepsilon^2}{16} \quad \text{in } \mathscr{U}_\delta, \ w \in U_R.$$

Outside $\mathscr{U}_\delta$, $\mathsf{d}^{2-2\varepsilon}$ is bounded from below by a positive constant. This proves the uniform upper bound in the second assertion. □

Lemma 3.9. *For each $w \in E_{\frac{1}{p}}$, $\left|\mathsf{d}^{2\varepsilon-1}\frac{d}{dx}\alpha_\varepsilon(H + w)\right| \sim \mathbf{1}$ in a δ-neighborhood $\mathscr{U}_\delta$ of Γ.*

Proof. Let $u = H + w$ and observe that

$$\frac{d}{dx}\alpha_\varepsilon(u) = -2\alpha_\varepsilon^2(u)\partial^{1-\varepsilon}u\frac{d}{dx}\partial^{1-\varepsilon}u.$$

An easy computation and Proposition 3.4 show that

$$\mathsf{d}^{2-\varepsilon}\left|\frac{d}{dx}\partial^{1-\varepsilon}H\right| \sim \mathbf{1}.$$

By (3.7), in a sufficiently small δ-neighborhood $\mathscr{U}_\delta$ of Γ, we have that

$$\mathsf{d}^{1-\varepsilon}|\partial^{1-\varepsilon}u| \sim \mathbf{1}, \qquad \mathsf{d}^{2-\varepsilon}\left|\frac{d}{dx}\partial^{1-\varepsilon}u\right| \sim \mathbf{1}.$$

In combination with Lemma 3.8, this yields

$$\left|\mathsf{d}^{2\varepsilon-1}\frac{d}{dx}\alpha_\varepsilon(u)\right| \sim \mathbf{1}, \quad \text{in } \mathscr{U}_\delta. \quad \square \tag{3.8}$$

Lemma 3.10. *There exists a constant C such that*

$$1/C < \operatorname{sign}(1-2\varepsilon)\mathsf{d}^{2\varepsilon}\frac{d^2}{dx^2}\alpha_\varepsilon(H) < C$$

in a δ-neighborhood $\mathscr{U}_\delta$ of Γ.

Proof. Direct computations show

$$\begin{aligned}\frac{d^2}{dx^2}\alpha_\varepsilon(H) &= \alpha_\varepsilon(H)^3|\partial^{1-\varepsilon}H|^2\left[6\left|\frac{d}{dx}\partial^{1-\varepsilon}H\right|^2 - 2\partial^{1-\varepsilon}H\frac{d^2}{dx^2}\partial^{1-\varepsilon}H\right]\\ &\quad - 2\alpha_\varepsilon(H)^3\left[\partial^{1-\varepsilon}H\frac{d^2}{dx^2}\partial^{1-\varepsilon}H + \left|\frac{d}{dx}\partial^{1-\varepsilon}H\right|^2\right].\end{aligned}$$

By Proposition 3.4, one can verify that

$$\mathsf{d}^{3-\varepsilon}\frac{d^2}{dx^2}\partial^{1-\varepsilon}H \sim \mathbf{1},$$

and

$$6\left|\frac{d}{dx}\partial^{1-\varepsilon}H\right|^2 - 2\partial^{1-\varepsilon}H\frac{d^2}{dx^2}\partial^{1-\varepsilon}H \sim 2(1-\varepsilon)(1-2\varepsilon)\mathsf{d}^{2\varepsilon-4}, \quad \text{near } \Gamma.$$

Note that, by the previous estimates and Lemma 3.9, it holds that

$$\mathsf{d}^{2\varepsilon}\alpha_\varepsilon(H)^3\left|\partial^{1-\varepsilon}H\frac{d^2}{dx^2}\partial^{1-\varepsilon}H + \left|\frac{d}{dx}\partial^{1-\varepsilon}H\right|^2\right| \sim \mathsf{d}^{2-2\varepsilon}.$$

Thus this term can be made arbitrarily small by shrinking $\mathscr{U}_\delta$. To sum up, in a sufficiently small δ-tubular neighborhood $\mathscr{U}_\delta$, we have

$$\operatorname{sign}(1-2\varepsilon)\mathsf{d}^{2\varepsilon}\frac{d^2}{dx^2}\alpha_\varepsilon(H) \sim |1-2\varepsilon|.$$

This completes the proof. $\square$

Lemma 3.11. *$\alpha_\varepsilon(H) \in BC_\pi^{2,2\varepsilon-2}(\mathsf{Q}\setminus\Gamma)$, and for each $w \in E_{\frac{1}{p}}$, we have*

$$\alpha_\varepsilon(H+w) \in BC_\pi^{1,2\varepsilon-2}(\mathsf{Q}\setminus\Gamma).$$

Moreover, for any $R > 0$,

$$[w \mapsto \alpha_\varepsilon(H + w)] \in C^\omega(B_R, BC_\pi^{1,2\varepsilon-2}(\mathsf{Q} \setminus \Gamma)),$$

where ω is the symbol for real analyticity.

Proof. By Lemma 3.3 and Proposition 3.4, we readily infer that

$$\alpha_\varepsilon(H) \in C_\pi^2(\mathsf{Q} \setminus \Gamma) \quad \text{and} \quad \alpha_\varepsilon(H + w) \in C_\pi^1(\mathsf{Q} \setminus \Gamma).$$

The rest of the proof for the first assertion follows from Lemmas 3.8–3.10.

By the estimates in Lemmas 3.8 and 3.9, we already knew that

$$\partial^{1-\varepsilon}(H + w) \in BC^{1,1-\varepsilon}(\mathsf{Q} \setminus \Gamma).$$

Lemma 2.5 implies that

$$\left[w \mapsto \frac{1}{\alpha_\varepsilon(H + w)}\right] \in C^\omega\left(E_{\frac{1}{p}}, BC_\pi^{1,2-2\varepsilon}(\mathsf{Q} \setminus \Gamma)\right).$$

The manifold $(\hat{\mathsf{M}}, \hat{g}) := (\mathsf{M}, g/\rho^2)$ has bounded geometry, and thus BC^k-function spaces are well defined. We denote these spaces by $BC^k(\hat{\mathsf{M}})$. Note that the space

$$BC_\pi^{1,0}(\mathsf{Q} \setminus \Gamma) \doteq \phi^* BC^1(\hat{\mathsf{M}}).$$

See [4, Section 4]. Applying [23, Proposition 6.3] to BC^k-functions and in view of Lemma 2.4, we can infer that

$$[w \mapsto \alpha_\varepsilon(H + w)] \in C^\omega(B_R, BC_\pi^{1,2\varepsilon-2}(\mathsf{Q} \setminus \Gamma))$$

for any $R > 0$. □

Lemma 3.12. *Assume that $w \in E_{\frac{1}{p}}$. Then for each k, there exists a sufficiently small δ_k-neighborhood $\mathscr{U}_{\delta_k}$ of Γ such that*

$$\|\alpha_\varepsilon(H) - \alpha_\varepsilon(H + w)\|_{BC^{1,2\varepsilon-2}(\bar{\mathscr{U}}_{\delta_k} \setminus \Gamma)} \le 1/k.$$

Proof. We have

$$\alpha_\varepsilon(H) - \alpha_\varepsilon(H + w) = \alpha_\varepsilon(H)\alpha_\varepsilon(H + w)\left[\partial^{1-\varepsilon}(2H + w)\right]\partial^{1-\varepsilon} w.$$

Thus, by Lemma 3.8, for some $C > 0$

$$\mathsf{d}^{2\varepsilon-2}|\alpha_\varepsilon(H) - \alpha_\varepsilon(H + w)| \le C\mathsf{d}^{1-\varepsilon}|\partial^{1-\varepsilon} w|.$$

This term can be made arbitrarily small by shrinking the neighborhood $\mathscr{U}_\delta$. The estimate for $\frac{d}{dx}\left[\alpha_\varepsilon(H) - \alpha_\varepsilon(H + w)\right]$ follows similarly by utilizing Lemmas 3.3, 3.8 and 3.9. □

We can now establish the following maximal regularity property for the operator $\mathscr{A}(w)$ for every $w \in E_{\frac{1}{p}}$.

Proposition 3.13. *Let $1 < p < \infty$ and ε satisfy (3.1) or (3.2). Then, for any $w \in E_{\frac{1}{p}}$, the operator*

$$\mathscr{A}(w) \in \mathcal{MR}_p(W_{p,\pi}^{2,\vartheta}(\mathsf{Q} \setminus \Gamma), L_{p,\pi}^{\vartheta+2\varepsilon}(\mathsf{Q} \setminus \Gamma)).$$

Proof. This theorem is a consequence of the work in [21,22]. We would like to refer the reader to these two papers for more details, and thus only necessary explanations will be pointed out here.

(i) For small $\delta > 0$, by [5, Theorem 1.6], $(\bar{\mathscr{U}}_\delta \setminus \Gamma, dx)$ is a singular manifold.

Lemmas 3.8–3.10 imply that

$$\alpha_\varepsilon(H)^{\frac{1}{2-2\varepsilon}} \in BC^{2,-1}(\bar{\mathscr{U}}_\delta \setminus \Gamma), \qquad \mathsf{d}^{-1}\alpha_\varepsilon(H)^{\frac{1}{2-2\varepsilon}} \sim \mathbf{1}.$$

Put $h = \operatorname{sign}(1-2\varepsilon)\log \alpha_\varepsilon(H)$. Then direct computations show

$$\left|\alpha_\varepsilon(H)^{\frac{1}{2-2\varepsilon}}\frac{d}{dx}h\right| = \left|\alpha_\varepsilon(H)^{\frac{2\varepsilon-1}{2-2\varepsilon}}\frac{d}{dx}\alpha_\varepsilon(H)\right| \sim \left|\mathsf{d}^{2\varepsilon-1}\frac{d}{dx}\alpha_\varepsilon(H)\right| \sim \mathbf{1},$$

via Lemma 3.9, and by Lemma 3.10

$$\alpha_\varepsilon(H)^{\frac{2\varepsilon}{2-2\varepsilon}}\frac{d}{dx}\left(\alpha_\varepsilon(H)\frac{d}{dx}h\right) = \operatorname{sign}(1-2\varepsilon)\alpha_\varepsilon(H)^{\frac{2\varepsilon}{2-2\varepsilon}}\frac{d^2}{dx^2}\alpha_\varepsilon(H) \sim \operatorname{sign}(1-2\varepsilon)\mathsf{d}^{2\varepsilon}\frac{d^2}{dx^2}\alpha_\varepsilon(H) \sim \mathbf{1}.$$

Therefore, the function h satisfies conditions $(\mathscr{H}_{2\varepsilon}1)$ and $(\mathscr{H}_{2\varepsilon}2)$ defined in [21, Section 5.1] with $\lambda = 2\varepsilon$ on $(\bar{\mathscr{U}}_\delta \setminus \Gamma, dx)$. This means that $(\bar{\mathscr{U}}_\delta \setminus \Gamma, dx)$ with $\alpha_\varepsilon(H)^{\frac{1}{2-2\varepsilon}}$ as a *singularity function* is a singular manifold satisfying property $\mathscr{H}_{2\varepsilon}$. The reader may refer to [21] for more details.

The proof of [21, Theorem 5.18] shows that the operator $-\mathscr{A}(0)$ generates an analytic contraction strongly continuous semigroup on $L_p^{2\varepsilon+\vartheta}(\bar{\mathscr{U}}_\delta \setminus \Gamma)$ with

$$D(\mathscr{A}(0)) \doteq \mathring{W}_p^{2,\vartheta}(\bar{\mathscr{U}}_\delta \setminus \Gamma), \quad 1 < p < \infty.$$

Here for $\mathfrak{F} \in \{BC, W_p\}$, $\mathring{\mathfrak{F}}^{s,\vartheta}(\bar{\mathscr{U}}_{\delta_k} \setminus \Gamma)$ is defined as the closure of $\mathcal{D}(\mathscr{U}_\delta \setminus \Gamma)$ in $\mathfrak{F}_\pi^{s,\vartheta}(\mathsf{Q} \setminus \Gamma)$. One can show that the semigroup $\{e^{-t\mathscr{A}(0)}\}_{t\geq 0}$ is positive by means of the same argument as in step (iii) of the proof for [22, Theorem 4.8].

(ii) Let

$$X_0(\delta) = L_p^{2\varepsilon+\vartheta}(\bar{\mathscr{U}}_\delta \setminus \Gamma), \qquad X_1(\delta) = \mathring{W}_p^{2,\vartheta}(\bar{\mathscr{U}}_\delta \setminus \Gamma).$$

Now, following exactly the same argument as in step (iv) and (4.14) of the proof for [22, Theorem 4.8], one concludes that

$$\mathscr{A}(0) \in \mathcal{RS}(X_0(\delta)) \quad \text{with } \phi^R_{\mathscr{A}(0)} < \pi/2.$$

Moreover, by the definition of $\mathscr{R}$-bound, it is easy to verify that for some $\theta > \pi/2$

$$\mathscr{R}\{\mu(\mu+\mathscr{A})^{-1} : \mu \in \Sigma_\theta\} \quad \text{is increasing in } \delta.$$

So is the norm $\|\mathscr{A}^{-1}(0)\|_{X_0(\delta),X_1(\delta)}$.

It follows from Lemmas 2.3 and 3.12 that, by shrinking δ, we can always make $\|(\mathscr{A}(w) - \mathscr{A}(0))\mathscr{A}^{-1}(0)\|_{\mathcal{L}(X_0(\delta))}$ arbitrarily small. As a direct consequence of the perturbation theorem of $\mathscr{R}$-sectorial operators, cf. [9, Proposition 4.2], we infer that

$$\mathscr{A}(w) \in \mathcal{RS}(X_0(\delta)) \quad \text{with } \phi^R_{\mathscr{A}(w)} < \pi/2.$$

The last step is to use a standard decomposition and gluing procedure as in step (v)–(vii) of the proof for [22, Theorem 4.8], and we can prove that for some $\omega \geq 0$

$$\omega + \mathscr{A}(w) \in \mathcal{RS}(L_{p,\pi}^{2\varepsilon+\vartheta}(\mathsf{Q} \setminus \Gamma)) \quad \text{with } \phi^R_{\mathscr{A}(w)} < \pi/2.$$

Then the assertion follows from [9, Theorem 4.4]. $\square$

Now we will apply Proposition 3.13 to proving existence and uniqueness of solutions to Eq. (2.1). We first consider the problem linearized in the initial datum H.

$$\begin{cases} \partial_t u - \operatorname{div}\big(\alpha_\varepsilon(H)\nabla u\big) = 0 & \text{in } \mathsf{Q}\times(0,\infty),\\ u & \text{periodic},\\ u(0) = H & \text{in } \mathsf{Q}. \end{cases} \tag{3.9}$$

Clearly, $u^* \equiv H$ solves (3.9).

Then we look at the nonlinear problem

$$\begin{cases} \partial_t u - \operatorname{div}\big(\alpha_\varepsilon(u+u^*)\nabla u\big) = 0 & \text{in } \mathsf{Q}\times(0,\infty),\\ u & \text{periodic},\\ u(0) = w_0 & \text{in } \mathsf{Q}. \end{cases} \tag{3.10}$$

Take $R>0$ so large that $w_0 \in B_R$, then by Lemmas 2.3, 2.5, 3.11 and Proposition 3.13,

$$\big[w \mapsto \operatorname{div}\big(\alpha_\varepsilon(H+w)\nabla\cdot\big)\big] \in C^\omega\big(B_R, \mathcal{MR}_p(E_1,E_0)\big). \tag{3.11}$$

Hence the condition in Theorem 2.10 is satisfied. The same theorem implies the existence of a unique solution

$$\tilde u \in \mathbb{E}_1(J) := L_p(J,E_1)\cap H^1_p(J,E_0)$$

to (3.10). We thus conclude that $\hat u = \tilde u + u^*$ is a solution to (2.1) with initial value $u_0 = H + w_0$.

We will show that $\hat u$ is indeed the unique solution in the class $\mathbb{E}_1(J)\oplus\mathbb{R}_\Gamma$, where

$$\mathbb{E}_1(J)\oplus\mathbb{R}_\Gamma := \big\{u\in L_{1,loc}(J\times(\mathsf{Q}\setminus\Gamma)) : u = u_1+u_2, u_1\in\mathbb{E}_1(J),\ u_2\in\mathbb{R}_\Gamma\big\}.$$

Note that by [7, Formula (2.1)] and (3.1) and (3.2)

$$\mathbb{E}_1(J)\hookrightarrow C\left(J, E_{\frac1p}\right), \quad \text{and} \quad \mathbb{R}_\Gamma\cap E_{\frac1p} = \{\mathbf{0}_{\mathsf{Q}\setminus\Gamma}\}.$$

Indeed, by Proposition 2.7, $E_{\frac1p}\hookrightarrow BC_\pi^{1,\vartheta+\frac{2\varepsilon+N}{p}}(\mathsf{Q}\setminus\Gamma)$. But $p > -\frac{N+2\varepsilon}{\vartheta}$ in (3.1) or $p>\frac{2N+2}{\varepsilon}$ in (3.2) implies that

$$u(x)\to 0 \quad \text{as } x\to\Gamma, \quad u\in BC_\pi^{1,\vartheta+\frac{2\varepsilon+N}{p}}(\mathsf{Q}\setminus\Gamma).$$

For any $u\in\mathbb{E}_1(J)\oplus\mathbb{R}_\Gamma$, we have thus a unique decomposition

$$u = u_1+u_2 \quad \text{with } u_1\in\mathbb{E}_1(J),\ u_2\in\mathbb{R}_\Gamma.$$

If $u\in\mathbb{E}_1(J)\oplus\mathbb{R}_\Gamma$ solves (2.1), by $u(0)=u_1(0)+u_2 = w_0+H$, we immediately infer that $u_2 = H$. Now the uniqueness of the solution to (3.10) implies $u_1=\tilde u$. The uniqueness of the solution to (2.1) in $\mathbb{E}_1(J)\oplus\mathbb{R}_\Gamma$ follows.

We are now ready to state the following well-posedness theorem for (2.1).

Theorem 3.14. *Assume that one of the following conditions holds*

$\varepsilon\in(0,\frac12)\cup(\frac12,1)$, $\vartheta\le -2$ *and* $p>\max\{\frac3\varepsilon, -\frac{1+2\varepsilon}{\vartheta}\}$, *or*
$\varepsilon\in(1-\frac{1}{2p},1)$, $\vartheta=-2\varepsilon$, *and* $p>\max\{\frac4\varepsilon,\frac92\}$.

Suppose that $\mathsf{Q}=[-1,1)$ *and that* H *is a piecewise constant function on* Q. *Let* Γ *be the discontinuity set of* H. *Then, given any* $u_0 = H+w_0$ *with*

$$w_0\in W_{p,\pi}^{2-2/p,\vartheta+\frac{2\varepsilon}{p}}(\mathsf{Q}\setminus\Gamma).$$

Eq. (2.1) *has a unique solution*

$$u \in L_p(J, W^{2,\vartheta}_{p,\pi}(\mathsf{Q} \setminus \Gamma)) \cap H^1_p(J, L^{\vartheta+2\varepsilon}_{p,\pi}(\mathsf{Q} \setminus \Gamma)) \oplus \mathbb{R}_\Gamma$$

for some $J := [0, T]$ *with* $T = T(u_0) > 0$. *Moreover,*

$$u \in C\left(J, W^{2-2/p,\vartheta+\frac{2\varepsilon}{p}}_{p,\pi}(\mathsf{Q} \setminus \Gamma)\right) \oplus \mathbb{R}_\Gamma.$$

3.2. Two dimensional case

In dimension two, the fractional gradient is defined again via Fourier series. For a periodic function u, $\nabla^{1-\varepsilon} u$ is defined as

$$\nabla^{1-\varepsilon} u := \mathcal{F}^{-1} \operatorname{diag}\{|k|^{-\varepsilon}\} \mathcal{F}\, |\nabla u|.$$

The choice of $|\nabla u|$ instead of ∇u is mainly for computational simplification.

Lemma 3.15. *For all* $u \in C^1_\pi(\mathsf{Q}^2)$

$$\nabla^{1-\varepsilon} u(x) = c_\varepsilon \int_{\mathsf{Q}^2} \frac{|\nabla u|(y)}{|x-y|^{2-\varepsilon}}\, dy + \int_{\mathsf{Q}^2} h_\varepsilon(x-y) |\nabla u|(y)\, dy$$

for some constant $c_\varepsilon > 0$ *and* $h_\varepsilon \in C^\infty$.

Proof. It follows from a proof similar to that of Lemma 3.2 and the two dimensional Poisson summation formula. □

Lemma 3.16.

$$\nabla^{1-\varepsilon} \in \mathcal{L}\left(E_{\frac{1}{p}}, BC^1_\pi(\mathsf{Q}^2)\right).$$

Proof. The proof is the same as that of Lemma 3.3. □

We are interested in initial data close to linear combinations of characteristic functions of disjoint bounded C^3-domains. Just like in the one dimensional case, we take a generic initial value function $H = \chi_\Omega$, where $\Omega \subset \mathring{\mathsf{Q}}^2$ is a bounded C^3-domain, and let $\Gamma = \partial\Omega$.

Since Ω is a set of finite perimeter, it is reasonable to take $|\nabla H| = \|\partial\Omega\|$. It is known that $\|\partial\Omega\| = \mathcal{H}^1 \llcorner \Gamma$; see [10, Section 5.1]. For any $\psi \in C^\infty_c(\mathsf{Q}^2)$,

$$\begin{aligned}
\langle \psi, \nabla^{1-\varepsilon} H\rangle &= \langle \mathcal{F}^{-1}|k|^{-\varepsilon}\mathcal{F}\psi, |\nabla H|\rangle \\
&= \int_\Gamma \int_{\mathsf{Q}^2} \left(\frac{c_\varepsilon}{|x-y|^{2-\varepsilon}} + h_\varepsilon(x-y)\right) \psi(x)\, dx\, d\mathcal{H}^1(y) \\
&= \int_{\mathsf{Q}^2} \psi(x) \int_\Gamma \left(\frac{c_\varepsilon}{|x-y|^{2-\varepsilon}} + h_\varepsilon(x-y)\right) d\mathcal{H}^1(y)\, dx
\end{aligned}$$

by Fubini's Theorem and Lemma 3.15, and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. So we have

$$\nabla^{1-\varepsilon} H(x) = \int_\Gamma \frac{c_\varepsilon}{|x-y|^{2-\varepsilon}}\, d\mathcal{H}^1(y) + \int_\Gamma h_\varepsilon(x-y)\, d\mathcal{H}^1(y), \quad x \in \mathsf{Q}^2 \setminus \Gamma.$$

Moreover, by its convolution definition, $\nabla^{1-\varepsilon} H \in C^\infty_\pi(\mathsf{Q}^2 \setminus \Gamma)$.

Proposition 3.17.

$$\nabla^{1-\varepsilon} H \in BC_{\pi}^{2,1-\varepsilon}(\mathsf{Q}^2 \setminus \Gamma),$$

and the following estimates hold in a δ-tubular neighborhood $\mathscr{U}_\delta$ of Γ

$$\mathsf{d}^{1-\varepsilon}\nabla^{1-\varepsilon}H \sim \mathbf{1}, \qquad \mathsf{d}^{2-\varepsilon}|\nabla\nabla^{1-\varepsilon}H| \sim \mathbf{1}, \qquad \mathsf{d}^{3-\varepsilon}|\Delta\nabla^{1-\varepsilon}H| \sim \mathbf{1},$$

along with

$$\operatorname{sign}(1-2\varepsilon)\mathsf{d}^{2\varepsilon}\Delta\alpha_\varepsilon(H) \sim \mathbf{1}.$$

Proof. Without loss of generality, we may assume that Ω is simply connected. More complicated situation can be treated similarly.

(i) Let $I(x) := \int_\Gamma \frac{1}{|x-y|^{2-\varepsilon}}\, d\mathcal{H}^1(y)$. To estimate $I(x)$ for those x inside a δ-tubular neighborhood $\mathscr{U}_\delta$ of Γ, we first note that there exists a diffeomorphism

$$\Lambda : \mathscr{U}_\delta \to \Gamma \times (-\delta,\delta) : \; x \mapsto (\Pi(x), d_\Gamma(x)),$$

where $\Pi(x)$ is the metric projection of x onto Γ and $d_\Gamma(x)$ is the signed distance from x to Γ. $d_\Gamma(x) < 0$ if x is in the interior of Γ.

$$\Lambda^{-1} : \Gamma \times (-\delta,\delta) \to \mathscr{U}_\delta : \; (\mathsf{p}, s) \mapsto \mathsf{p} + s\nu_\Gamma(\mathsf{p}),$$

where ν_Γ denotes the outer normal of Γ. Λ and Λ^{-1} are C^2-continuous. For every $x \in \mathscr{U}_\delta$, we pick a coordinate chart, O_x, around $\Pi(x)$ and chart maps ψ_x, φ_x such that

$$\psi_x : \mathsf{O}_x \to (-1,1) \quad \text{with } \varphi_x = \psi_x^{-1} \quad \text{and} \quad \psi_x(\Pi(x)) = 0.$$

Moreover, $\varphi_x^* g_N|_\Gamma \sim g_1$, the one dimensional Euclidean metric, uniformly in x.

To estimate $I(x)$ for $x \in \mathscr{U}_\delta$, first notice that

$$\int_{\mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\, d\mathcal{H}^1(y) \sim \int_{-1}^{1} \frac{1}{(y^2+z^2)^{\frac{2-\varepsilon}{2}}}\, dy,$$

where $z = d_\Gamma(x)$. The Lipschitz constant in this equivalence is independent of x. Without loss of generality, we assume that $z > 0$ and $\delta < 1$; then

$$\begin{aligned}
\int_{-1}^{1} \frac{dy}{(y^2+z^2)^{\frac{2-\varepsilon}{2}}} &= \frac{1}{z^{1-\varepsilon}} \int_{-1/z}^{1/z} \frac{dy}{(1+y^2)^{\frac{2-\varepsilon}{2}}} \\
&= \frac{2}{z^{1-\varepsilon}} \left(\int_0^1 \frac{dy}{(1+y^2)^{\frac{2-\varepsilon}{2}}} + \int_1^{1/z} \frac{dy}{(1+y^2)^{\frac{2-\varepsilon}{2}}} \right) \\
&\sim \frac{1}{z^{1-\varepsilon}}
\end{aligned}$$

for δ sufficiently small. On the other hand, by choosing δ possibly even smaller, we can always make

$$\int_{\Gamma\setminus \mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\, d\mathcal{H}^1(y) < \frac{1}{2}\int_{\mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\, d\mathcal{H}^1(y).$$

(ii) To estimate $|\nabla I(x)|$, we first compute

$$\nabla I(x) = (\varepsilon - 2)\int_\Gamma \frac{x-y}{|x-y|^{4-\varepsilon}}\, d\mathcal{H}^1(y).$$

By the above estimates, it is not hard to see that, in order to bound

$$\Big| \int_{\mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\, d\mathcal{H}^1(y) \Big|,$$

it suffices to look at

$$\int_{-1}^{1} \frac{z\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}} \quad \text{and} \quad \int_{-1}^{1} \frac{|y|\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}.$$

A similar computation as above yields

$$\int_{-1}^{1} \frac{z\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}, \int_{-1}^{1} \frac{|y|\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}} \sim \frac{1}{z^{2-\varepsilon}}.$$

Again by choosing δ small enough, we can always make

$$\Big|\int_{\Gamma\setminus \mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big| < \frac{1}{2}\Big|\int_{\mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big|.$$

(iii) Since

$$\Delta\nabla^{1-\varepsilon}H(x) = (\varepsilon-2)^2\int_{\Gamma} \frac{1}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y) + \tilde{h}_\varepsilon(x),$$

where $\tilde{h}_\varepsilon \in C^\infty$, the estimate for $\Delta\nabla^{1-\varepsilon}H$ follows in an analogous way. Combining everything together, it is clear that

$$\mathsf{d}^{1-\varepsilon}(x)\nabla^{1-\varepsilon}H(x) \sim \mathbf{1}, \qquad \mathsf{d}^{2-\varepsilon}(x)|\nabla\nabla^{1-\varepsilon}H(x)| \sim \mathbf{1}, \qquad \mathsf{d}^{3-\varepsilon}(x)|\Delta\nabla^{1-\varepsilon}H(x)| \sim \mathbf{1}$$

hold for all $x \in \mathscr{U}_\delta$. The fact that $\nabla^{1-\varepsilon}H \in BC^{2,1-\varepsilon}_{\pi}(\mathsf{Q}^2 \setminus \Gamma)$ follows from these estimates and the definition of weighted BC^k-spaces.

(iv) As in Lemma 3.10, direct computations show that

$$\begin{aligned}\Delta\alpha_\varepsilon(H) &= \alpha_\varepsilon(H)^3|\nabla^{1-\varepsilon}H|^2\big[6|\nabla\nabla^{1-\varepsilon}H|^2 - 2\nabla^{1-\varepsilon}H\Delta\nabla^{1-\varepsilon}H\big]\\ &\quad - 2\alpha_\varepsilon(H)^3\big[\nabla^{1-\varepsilon}H\Delta\nabla^{1-\varepsilon}H + |\nabla\nabla^{1-\varepsilon}H|^2\big].\end{aligned}$$

Again as in Lemma 3.10, we only need to estimate

$$\begin{aligned}&6|\nabla\nabla^{1-\varepsilon}H|^2 - 2\nabla^{1-\varepsilon}H\Delta\nabla^{1-\varepsilon}H\\ &\quad\sim \Big[3\Big|\int_{\Gamma} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big|^2 - \int_{\Gamma} \frac{1}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\int_{\Gamma} \frac{1}{|x-y|^{2-\varepsilon}}\,d\mathcal{H}^1(y)\Big].\end{aligned}$$

To estimate the right hand side, as in (i)–(iii), it suffices to look at $x \in \mathscr{U}_\delta$ and $y \in \mathsf{O}_x$. We need a more precise estimate than those in (i)–(iii), i.e.

$$\begin{aligned}&3\Big|\int_{\mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big|^2 - \int_{\mathsf{O}_x} \frac{1}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\int_{\mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\,d\mathcal{H}^1(y)\\ &= 3\Big(\int_{-s}^{s} \frac{z\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}J(y)\Big)^2 + 3\Big(\int_{-s}^{s} \frac{y\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}J(y)\Big)^2\\ &\quad - \int_{-s}^{s} \frac{1}{(y^2+z^2)^{\frac{2-\varepsilon}{2}}}J(y)\,dy\int_{-s}^{s} \frac{1}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}J(y)\,dy,\end{aligned}$$

where $J(y) \in (K(1-\mu), K(1+\mu))$ for some $K, \mu > 0$. μ is independent of x and can be chosen arbitrarily small by first shrinking O_x, or equivalently s, and then $\mathscr{U}_\delta$. Therefore, for each μ_0, by shrinking O_x and $\mathscr{U}_\delta$, we have that

$$
\begin{aligned}
&3\Big|\int_{\mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big|^2 - \int_{\mathsf{O}_x} \frac{1}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\int_{\mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\,d\mathcal{H}^1(y)\\
&\le 2K_0\Big\{(3+\mu_0)\Big(\int_0^s \frac{z\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}\Big)^2 + 2\mu(3+\mu_0)\Big(\int_0^s \frac{y\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}\Big)^2\\
&\quad - \int_0^s \frac{dy}{(y^2+z^2)^{\frac{2-\varepsilon}{2}}}\int_0^s \frac{dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}\Big\}\\
&= \frac{2K_0}{z^{4-2\varepsilon}}\Big\{(3+\mu_0)\Big(\int_0^{\frac{s}{z}} \frac{dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}}\Big)^2 + 2\mu(3+\mu_0)\Big(\int_0^{\frac{s}{z}} \frac{y\,dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}}\Big)^2\\
&\quad - \int_0^{\frac{s}{z}} \frac{dy}{(y^2+1)^{\frac{2-\varepsilon}{2}}}\int_0^{\frac{s}{z}} \frac{dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}}\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&3\Big|\int_{\mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big|^2 - \int_{\mathsf{O}_x} \frac{1}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\int_{\mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\,d\mathcal{H}^1(y)\\
&\ge K_0\Big\{(3-\mu_0)\Big(\int_{-s}^s \frac{z\,dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}\Big)^2 - \int_{-s}^s \frac{dy}{(y^2+z^2)^{\frac{2-\varepsilon}{2}}}\int_{-s}^s \frac{dy}{(y^2+z^2)^{\frac{4-\varepsilon}{2}}}\Big\}\\
&= \frac{2K_0}{z^{4-2\varepsilon}}\Big\{(3-\mu_0)\Big(\int_0^{\frac{s}{z}} \frac{dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}}\Big)^2 - \int_0^{\frac{s}{z}} \frac{dy}{(y^2+1)^{\frac{2-\varepsilon}{2}}}\int_0^{\frac{s}{z}} \frac{dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}}\Big\}
\end{aligned}
$$

for some $K_0 > 0$. Recall that μ, and thus $2\mu(3+\mu_0)$, can be made arbitrarily small, and note that once O_x, i.e. s, is fixed, s/z can be made arbitrarily large by further shrinking $\mathscr{U}_\delta$. Therefore, we have

$$
\begin{aligned}
&\mathsf{d}^{4-2\varepsilon}\Big[3\Big|\int_{\mathsf{O}_x} \frac{x-y}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\Big|^2 +\\
&\quad - \int_{\mathsf{O}_x} \frac{1}{|x-y|^{4-\varepsilon}}\,d\mathcal{H}^1(y)\int_{\mathsf{O}_x} \frac{1}{|x-y|^{2-\varepsilon}}\,d\mathcal{H}^1(y)\Big] \sim \mathbf{1},
\end{aligned}
$$

as long as

$$
3\int_0^\infty \frac{dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}} - \int_0^\infty \frac{dy}{(y^2+1)^{\frac{2-\varepsilon}{2}}} \neq 0. \tag{3.12}
$$

One verifies that

$$
3\int_0^\infty \frac{dy}{(y^2+1)^{\frac{4-\varepsilon}{2}}} - \int_0^\infty \frac{dy}{(y^2+1)^{\frac{2-\varepsilon}{2}}} = \frac{3}{2}B\left(\frac{1}{2}, \frac{3-\varepsilon}{2}\right) - \frac{1}{2}B\left(\frac{1}{2}, \frac{1-\varepsilon}{2}\right),
$$

where $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ is the Beta function. The right hand side equals zero iff $\varepsilon = 1/2$. Thus, we conclude that for all $\varepsilon \neq 1/2$

$$
\operatorname{sign}(1-2\varepsilon)\mathsf{d}^{2\varepsilon}\Delta\alpha_\varepsilon(H) \sim \mathbf{1}
$$

in a sufficiently small δ-tubular neighborhood $\mathscr{U}_\delta$ of Γ. □

Recall that $\mathbb{R}_\Gamma$ denotes the set of all functions that are constants in each connected component of $\mathsf{Q}^2 \setminus \Gamma$. Now, combining Lemma 3.16, Proposition 3.17, and an argument analogous to the one used in the one dimensional case, we obtain the following proposition.

Proposition 3.18. *Let $1 < p < \infty$ and ε satisfy (3.1) or (3.2). Then, for each $w \in E_{\frac{1}{p}}$, the operator*

$$
\mathscr{A}(w) \in \mathcal{MR}_p(W^{2,\vartheta}_{p,\pi}(\mathsf{Q}^2 \setminus \Gamma), L^{\vartheta+2\varepsilon}_{p,\pi}(\mathsf{Q}^2 \setminus \Gamma)).
$$

Proof. As in the proof for Proposition 3.13, we put $h = \mathrm{sign}(1-2\varepsilon)\log\alpha_\varepsilon(H)$. Easy computations show

$$\left|\alpha_\varepsilon(H)^{\frac{1}{2-2\varepsilon}}\nabla h\right| \sim \left|\mathsf{d}^{2\varepsilon-1}\frac{d}{dx}\alpha_\varepsilon(H)\right| \sim \mathbf{1},$$

and

$$\alpha_\varepsilon(H)^{\frac{2\varepsilon}{2-2\varepsilon}}\mathrm{div}(\alpha_\varepsilon(H)\nabla h) \sim \mathrm{sign}(1-2\varepsilon)\mathsf{d}^{2\varepsilon}\Delta\alpha_\varepsilon(H) \sim \mathbf{1},$$

near Γ. Then the rest of the proof follows in the same way as that for Proposition 3.13. □

The following theorem concerning the local wellposedness of Eq. (2.1) in two space dimensions follows.

Theorem 3.19. *Assume that one of the following conditions holds*

$\varepsilon \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $\vartheta \le -2$ *and* $p > \max\{\frac{4}{\varepsilon}, -\frac{2+2\varepsilon}{\vartheta}\}$ *or*
$\varepsilon \in (1-\frac{1}{2p}, 1)$, $\vartheta = -2\varepsilon$, *and* $p > \max\{\frac{6}{\varepsilon}, \frac{13}{2}\}$.

Suppose that H is a linear combination of characteristic functions of disjoint C^3-domains Ω_i in $\mathring{\mathsf{Q}}^2$. Let $\Gamma = \cup_i \partial\Omega_i$. Given any $u_0 = H + w_0$ with

$$w_0 \in W_{p,\pi}^{2-2/p,\vartheta+\frac{2\varepsilon}{p}}(\mathsf{Q}^2 \setminus \Gamma).$$

Eq. (2.1) has a unique solution

$$u \in L_p(J, W_{p,\pi}^{2,\vartheta}(\mathsf{Q}^2 \setminus \Gamma)) \cap H_p^1(J, L_{p,\pi}^{\vartheta+2\varepsilon}(\mathsf{Q}^2 \setminus \Gamma)) \oplus \mathbb{R}_\Gamma$$

for some $J := [0, T]$ with $T = T(u_0) > 0$. Moreover,

$$u \in C\left(J, W_{p,\pi}^{2-2/p,\vartheta+\frac{2\varepsilon}{p}}(\mathsf{Q}^2 \setminus \Gamma)\right) \oplus \mathbb{R}_\Gamma.$$

4. Global existence

In this section, we focus on the case (3.2)

$$\vartheta = -2\varepsilon, \qquad p > \max\left\{\frac{2N+2}{\varepsilon}, \frac{4N+5}{2}\right\}, \quad \varepsilon > 1 - \frac{1}{2p}$$

and prove global existence of the solutions to (2.1) to initial data close enough to an equilibrium. Note that (3.2) implies the necessary condition $\varepsilon > 1/2$ in the sequel, and this is why only (3.2) is considered in this section.

In [11, Proposition 6], the first author proves that characteristic functions of smooth domains Ω are stationary solutions for (2.1). While in that article, the submanifold $\Gamma = \partial\Omega$ is required to be smooth, lower regularity, e.g. C^3-regularity, suffices.

Proposition 4.1. *Linear combinations of characteristic functions of disjoint C^3-domains Ω_i in $\mathring{\mathsf{Q}}^N$ are stationary solutions to (2.1).*

We define

$$P: \ W_{p,\pi}^{2,-2\varepsilon}(\mathsf{Q}^N \setminus \Gamma) \to L_{p,\pi}(\mathsf{Q}^N \setminus \Gamma): \ u \mapsto \mathrm{div}\Big(\frac{1}{1+|\nabla^{1-\varepsilon}(H+u)|^2}\nabla u\Big),$$

where $H = \chi_\Omega$ for some C^3-domain $\Omega \subset \mathring{\mathsf{Q}}^N$. The discussions in the previous section (cf. (3.11)) show that

$$P \in C^\omega(W_{p,\pi}^{2,-2\varepsilon}(\mathsf{Q}^N \setminus \Gamma), L_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)).$$

Let

$$\mathscr{A}_{\alpha_\varepsilon} u := \operatorname{div}(\alpha_\varepsilon \nabla u), \qquad \alpha_\varepsilon := \frac{1}{1+|\nabla^{1-\varepsilon} H|^2}.$$

Note that $\alpha_\varepsilon \sim \mathsf{d}^{2-2\varepsilon}$. Denote the Fréchet derivative of P at 0 by $\partial P(0)$. Then an easy computation shows that $\partial P(0) = \mathscr{A}_{\alpha_\varepsilon}$. Consider the following abstract linear equation.

$$\begin{cases} \partial_t u - \mathscr{A}_{\alpha_\varepsilon} u = 0 & \text{in } \mathsf{Q}^N \times (0,\infty), \\ u & \text{periodic}, \\ u(0) = u_0 & \text{in } \mathsf{Q}^N. \end{cases} \tag{4.1}$$

We can associate with $\mathscr{A}_{\alpha_\varepsilon}$ a form operator $\mathfrak{a}$ with $D(\mathfrak{a}) = \mathring{H}^1_{\alpha_\varepsilon,\pi}(\mathsf{Q}^N \setminus \Gamma)$, defined by

$$\mathfrak{a}(u,v) = \int_{\mathsf{Q}^N} \alpha_\varepsilon (\nabla u | \nabla v)\, dx,$$

for $u, v \in D(\mathfrak{a})$. Here $\mathring{H}^1_{\alpha_\varepsilon,\pi}(\mathsf{Q}^N \setminus \Gamma)$ is the closure of $\mathcal{D}_\pi(\mathsf{Q}^N \setminus \Gamma)$, where $\mathcal{D}_\pi(\mathsf{Q}^N \setminus \Gamma) = \phi^* \mathcal{D}(\mathsf{M})$, with respect to the norm $\|\cdot\|_{\alpha_\varepsilon}$,

$$\|u\|_{\alpha_\varepsilon} = (\|u\|_2^2 + \|\sqrt{\alpha_\varepsilon} \nabla u\|_2^2)^{1/2}$$

with $\|\cdot\|_2$ being the norm of $L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$.

Lemma 4.2. (i) *The embedding $D(\mathfrak{a}) \hookrightarrow L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$ is compact.*
(ii) *Any function $u \in D(\mathfrak{a})$ admits a trace $\gamma_\Gamma(u) = 0$ a.e. on Γ.*
(iii) *It holds that*

$$\|u\|_2 \le C \|\sqrt{\alpha_\varepsilon}\, \nabla u\|_2, \quad u \in D(\mathfrak{a}),$$

where $\|\cdot\|_2$ is the norm of $L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$.

Proof. (i) Since $\alpha_\varepsilon \sim \mathsf{d}^{2-2\varepsilon}$ and $\varepsilon > 1/2$, there exists an $q > 1$ such that

$$\int_{\mathsf{Q}^N \setminus \Gamma} \frac{1}{\alpha_\varepsilon^q(x)}\, dx < \infty.$$

Then one has that $|\nabla u| \in W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$ for some small enough $s > 0$ since

$$\begin{aligned} \int_{\mathsf{Q}^N \setminus \Gamma} |\nabla u|^{1+s}\, dx &\le \int_{\mathsf{Q}^N \setminus \Gamma} \left(\frac{\sqrt{\alpha_\varepsilon(x)}}{\sqrt{\alpha_\varepsilon(x)}} \right)^{1+s} |\nabla u|^{1+s}\, dx \\ &\le \left(\int_{\mathsf{Q}^N \setminus \Gamma} \alpha_\varepsilon(x)^{-\frac{1+s}{1-s}}\, dx \right)^{\frac{1-s}{2}} \left(\int_{\mathsf{Q}^N \setminus \Gamma} \alpha_\varepsilon(x) |\nabla u(x)|^2\, dx \right)^{\frac{1+s}{2}} < \infty \end{aligned} \tag{4.2}$$

provided that $\frac{1+s}{1-s} < q$, which is always possible for a small enough s. This shows that $u \in W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$. The claim therefore follows from the compactness part of Sobolev embedding theorem. This is obvious for $N = 1$. For $N = 2$, it follows observing that $2 < (1+s)^* = \frac{N(1+s)}{N-1-s}$ is valid as long as $N < 2\frac{1+s}{1-s}$. The latter is always the case for $N = 2$.

(ii) Inequality (4.2) implies that on each component Ω_i of $\mathsf{Q}^N \setminus \Gamma$,

$$D(\mathfrak{a}) \hookrightarrow W^1_{1+s}(\Omega_i)$$

for some $s > 0$ small. By the well known trace theorem,

$$\gamma_\Gamma \in \mathcal{L}\Big(W^1_{1+s}(\Omega_i), W^{1-\frac{1}{1+s}}_{1+s}(\partial \Omega_i) \Big).$$

Therefore the trace operator is well-defined on $D(\mathfrak{a})$ and

$$\gamma_\Gamma \in \mathcal{L}\Big(D(\mathfrak{a}), W_{1+s}^{1-\frac{1}{1+s}}(\partial\Omega_i)\Big) \tag{4.3}$$

on each connected component of $\mathsf{Q}^N \setminus \Gamma$. By the density of $\mathcal{D}_\pi(\mathsf{Q}^N \setminus \Gamma)$ in $D(\mathfrak{a})$, we can take a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{D}_\pi(\mathsf{Q}^N \setminus \Gamma)$ converging to u in $D(\mathfrak{a})$. Since $\gamma_\Gamma(u_k) = 0$, we conclude from (4.3) that $\gamma_\Gamma(u) = 0$ as well.

(iii) Given any $u \in D(\mathfrak{a})$, it follows from (4.2) that $u \in W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$ with s small enough, and by the trace lemma, we have $\gamma_\Gamma(u) = 0$. So we can apply the Poincaré inequality for $W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$ on each connected component of $\mathsf{Q}^N \setminus \Gamma$ to u, which yields

$$\|u\|_{L_{1+s,\pi}(\mathsf{Q}^N\setminus\Gamma)} \le C\|\nabla u\|_{L_{1+s,\pi}(\mathsf{Q}^N\setminus\Gamma)}.$$

In view of the embedding $W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma) \hookrightarrow L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$ and (4.2), it holds that

$$\|u\|_2 \le C\|u\|_{W^1_{1+s,\pi}(\mathsf{Q}^N\setminus\Gamma)} \le C\|\nabla u\|_{L_{1+s,\pi}(\mathsf{Q}^N\setminus\Gamma)} \le C\|\sqrt{\alpha_\varepsilon}\,\nabla u\|_2. \quad \square$$

Proposition 4.3. $\mathfrak{a}$ *is continuous and* $D(\mathfrak{a})$*-coercive. More precisely,*

(i) *(Continuity) there exists some constant C such that for all $u, v \in D(\mathfrak{a})$*

$$|\mathfrak{a}(u,v)| \le C\|u\|_{D(\mathfrak{a})}\|v\|_{D(\mathfrak{a})}.$$

(ii) *($D(\mathfrak{a})$-Coercivity) There is some C such that for any $u \in D(\mathfrak{a})$*

$$\mathrm{Re}(\mathfrak{a}(u,u)) \ge C\|u\|^2_{D(\mathfrak{a})}.$$

Proof. (i)

$$\begin{aligned}|\mathfrak{a}(u,v)| &= \left|\int_{\mathsf{Q}^N} \alpha_\varepsilon(\nabla u|\nabla v)\,dx\right| \\ &= \int_{\mathsf{Q}^N} |(\sqrt{\alpha_\varepsilon}\,\nabla u|\sqrt{\alpha_\varepsilon}\,\nabla v)|\,dx \\ &\le \|u\|_{D(\mathfrak{a})}\|v\|_{D(\mathfrak{a})}.\end{aligned}$$

The last step follows from Hölder inequality and $|(a|b)| \le |a|\,|b|$.

(ii) It is a direct consequence of Lemma 4.2(iii) that

$$\mathrm{Re}(\mathfrak{a}(u,u)) = \mathfrak{a}(u,v) = \int_{\mathsf{Q}^N} \alpha_\varepsilon|\nabla u|^2\,dx \ge C\|u\|_{D(\mathfrak{a})}. \quad \square$$

Proposition 4.3 shows that $\mathfrak{a}$ with $D(\mathfrak{a})$ is densely defined, sectorial and closed on $L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$. By [15, Theorems VI.2.1, IX.1.24], we can find an associated operator T such that $-T$ generates a strongly continuous analytic semigroup of contractions on $L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$, i.e. satisfying $\|e^{-tT}\|_{\mathcal{L}(L_{2,\pi}(\mathsf{Q}^N\setminus\Gamma))} \le 1$ for all $t \ge 0$. Its domain is given by

$$D(T) := \big\{u \in D(\mathfrak{a}) : \exists! v \in L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma) \text{ s.t. } \mathfrak{a}(u,\phi) = \langle v,\phi\rangle,\ \forall \phi \in D(\mathfrak{a})\big\}$$

and $Tu = v$; $D(T)$ is a core of $\mathfrak{a}$. The operator T is unique in the sense that there exists only one operator satisfying

$$\mathfrak{a}(u,v) = \langle Tu, v\rangle, \quad u \in D(T),\, v \in D(\mathfrak{a}).$$

We have proved that

$$W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma) \overset{d}{\hookrightarrow} D(\mathfrak{a}) \overset{d}{\hookrightarrow} W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma).$$

Here $W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$ is the closure of $\mathcal{D}_\pi(\mathsf{Q}^N \setminus \Gamma)$ in $W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$. Then we can uniquely extend $\mathscr{A}_{\alpha_\varepsilon}$, which is originally defined on $W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$ as in Section 3, to $W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$. Now $\mathscr{A}_{\alpha_\varepsilon}$ can be defined on $W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma)$ by

$$\langle \mathscr{A}_{\alpha_\varepsilon} u, v\rangle = -\mathfrak{a}(u,v), \quad u \in W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma),\ v \in \mathcal{D}_\pi(\mathsf{Q}^N \setminus \Gamma)$$

and $\mathscr{A}_{\alpha_\varepsilon} \in \mathcal{L}(W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma), (W^1_{1+s,\pi}(\mathsf{Q}^N \setminus \Gamma))')$. Restricted onto $D(\mathfrak{a})$ and by a density argument, this yields that for any $u,v \in D(\mathfrak{a})$

$$\langle \mathscr{A}_{\alpha_\varepsilon} u, v\rangle = -\mathfrak{a}(u,v),$$

and thus

$$|\langle \mathscr{A}_{\alpha_\varepsilon} u, v\rangle| \le C\|u\|_{D(\mathfrak{a})}\|v\|_{D(\mathfrak{a})},$$

which implies that $\mathscr{A}_{\alpha_\varepsilon} \in \mathcal{L}(D(\mathfrak{a}), (D(\mathfrak{a}))')$. Since it holds that

$$\mathscr{A}_{\alpha_\varepsilon} \in \mathcal{L}(W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma), L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma))$$

supported by Lemmas 2.3 and 2.5, we further have that, for any $u \in W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$ and $v \in D(\mathfrak{a})$,

$$|\mathfrak{a}(u,v)| = |\langle \mathscr{A}_{\alpha_\varepsilon} u, v\rangle| \le \|\mathscr{A}_{\alpha_\varepsilon} u\|_2 \|v\|_2 \le C\|u\|_{2,2;-2\varepsilon}\|v\|_2.$$

It is known that a function $u \in D(T)$ iff $u \in D(\mathfrak{a})$ and

$$|\mathfrak{a}(u,v)| \le C\|v\|_2, \quad v \in D(\mathfrak{a}).$$

Therefore, we conclude that

$$T = \mathscr{A}_{\alpha_\varepsilon}|_{D(T)} \quad \text{and} \quad W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma) \subset D(T).$$

On the other hand, choosing $w = 0$ in Proposition 3.13 yields

$$\mathscr{A}_{\alpha_\varepsilon} \in \mathcal{MR}_p(W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma), L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)), \quad 1 < p < \infty.$$

It is well known, see e.g. [19, Proposition 1.2], that this implies the existence of some $\omega \ge 0$ such that

$$\omega + \mathscr{A}_{\alpha_\varepsilon} \in \mathcal{L}\mathrm{is}(W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma), L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)) \cap \mathcal{S}(L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma))$$

with spectral angle $\phi_{\omega+\mathscr{A}_{\alpha_\varepsilon}} < \pi/2$.

Due to well-known results of semigroup theory, we know that for the same ω as above

$$\omega + \mathscr{A}_{\alpha_\varepsilon} \in \mathcal{L}\mathrm{is}(D(T), L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)),$$

from which we infer right away that

$$D(T) \doteq W^{2,-2\varepsilon}_{2,\pi}(\mathsf{Q}^N \setminus \Gamma).$$

By standard real analysis knowledge, we know that $u \in D(\mathfrak{a})$ implies the validity of $(|u|-1)^+ \mathrm{sign}\, u \in D(\mathfrak{a})$ and that

$$\nabla\big[(|u|-1)^+ \mathrm{sign}\, u\big] = \begin{cases} \nabla u, & |u| > 1; \\ 0, & |u| \le 1. \end{cases}$$

Here it is understood that

$$\operatorname{sign} u := \begin{cases} u/|u|, & u \neq 0; \\ 0, & u = 0. \end{cases}$$

Now it is clear that

$$\mathrm{Re}\big[\mathfrak{a}(u, (|u|-1)^{+}\operatorname{sign} u)\big] \geq 0.$$

By [17, Theorem 2.7], the semigroup $\{e^{-t\mathscr{A}_{\alpha_\varepsilon}}\}_{t\geq 0}$ is L_∞-contractive, or more precisely,

$$\|e^{-t\mathscr{A}_{\alpha_\varepsilon}} u\|_\infty \leq \|u\|_\infty, \quad t \geq 0,\ u \in L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma) \cap L_{\infty,\pi}(\mathsf{Q}^N \setminus \Gamma).$$

We can then follow a well-known argument, see [8, Chapter 1.4], to prove that for each $1 < p < \infty$, $\{e^{-t\mathscr{A}_{\alpha_\varepsilon}}\}_{t\geq 0}$ can be extended to a strongly continuous analytic semigroup of contractions on $L_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$. Then we can determine the domain for this semigroup by the same argument used previously for the semigroup on $L_{2,\pi}(\mathsf{Q}^N \setminus \Gamma)$. In sum, we can prove the following assertion.

Lemma 4.4. *$-\mathscr{A}_{\alpha_\varepsilon}$ generates a strongly continuous analytic semigroup of contractions on $L_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$ with domain $W^{2,-2\varepsilon}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$ for all $1 < p < \infty$.*

Now we apply the form operator method to the operator $\mathscr{A}_{\alpha_\varepsilon} - \omega$ for some sufficiently small positive ω. By Lemma 4.2(iii), we infer that Proposition 4.3 still holds true for $\mathscr{A}_{\alpha_\varepsilon} - \omega$ with ω small. Then we can follow the above argument step by step and prove the same contraction semigroup property for $\mathscr{A}_{\alpha_\varepsilon} - \omega$ as in Lemma 4.4. This immediately gives a spectral bound for $\mathscr{A}_{\alpha_\varepsilon}$.

Lemma 4.5. $\sup\big\{\mathrm{Re}(\mu) : \mu \in \sigma(-\mathscr{A}_{\alpha_\varepsilon})\big\} < 0.$

The (exponential) asymptotic stability of the stationary solution H now follows from well-known linearized stability results.

Theorem 4.6. *Assume that $\mathsf{Q}^N = [-1,1)^N$ with $N = 1, 2$ and*

$$\varepsilon \in \left(1 - \frac{1}{2p}, 1\right), \qquad p > \max\left\{\frac{2N+2}{\varepsilon}, \frac{4N+5}{2}\right\}.$$

Suppose that Γ is a C^3-submanifold in $\mathring{\mathsf{Q}}^N$. Let H be a component-wise constant function on $\mathsf{Q}^N \setminus \Gamma$. Then H is a stationary solution to (2.1) and attracts all solutions which are initially $W^{2-\frac{2}{p},\frac{2\varepsilon(1-p)}{p}}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$ close to H.

More precisely, if the initial datum satisfies

$$u_0 = H + w_0 \quad \text{with } w_0 \in W^{2-\frac{2}{p},\frac{2\varepsilon(1-p)}{p}}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$$

and $\|w_0\|_{2-\frac{2}{p},p;\frac{2\varepsilon(1-p)}{p}}$ sufficiently small, then the solution u to (2.1) converges to H exponentially fast in $W^{2-\frac{2}{p},\frac{2\varepsilon(1-p)}{p}}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$-topology, in particular, in $C^1(\mathsf{Q}^N)$-topology.

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