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Hopf Bifurcation in a Scalar Reaction Diffusion Equation

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On the assumption of separated boundary conditions autonomous scalar reaction-diffusion equations do not admit periodic orbits. The relevance of the assumption of separatedness is shown by giving an example of non separated boundary conditions for which Hopf bifurcation occurs. The example is a model of a simple thermostat.

1. INTRODUCTION

This research was inspired by the following passage in N. Wiener's monograph Cybernetics [10, Chapter 4, p. 96], where an ordinary thermostat is used as a simple example of a feedback chain:

> [...] There is a setting for the desired room temperature: and if the actual temperature of the house is below this, an apparatus is actuated which opens the damper, or increases the flow of fuel oil, and brings the temperature of the house up to the desired level. If, on the other hand, the temperature of the house exceeds the desired level, the dampers are turned off or the flow of fuel oil is slackened or interrupted. In this way the temperature of the house is kept approximately at a steady level. Note that the constancy of this level depends on the good design of the thermostat, and that a badly designed thermostat may send the temperature of the house into violent oscillations [...].

This paper is devoted to providing and discussing a model capable of explaining the cause of the oscillatory behaviour predicted by Wiener. We propose the following model of a rudimentary thermostat.

$$\begin{cases} u_t - u_{xx} = 0, & (x, t) \in (0, \pi) \times (0, \infty), \\ u_x(0, t) = \tanh(\beta u(\pi, t)), & t \in (0, \infty), \\ u_x(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = u_0(x), & x \in (0, \pi). \end{cases} \tag{1}$$

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This reaction-diffusion equation describes heat diffusion in a one dimensional wire of length π. At the right end $x=\pi$ we assume no-flux boundary conditions whereas the amount of cooling or heating at the other end is a nonlinear function of the temperature at $x=\pi$. We are thus considering nonlocal, or non-separated, boundary conditions. The nonlinearity $\tanh(\cdot)$ is chosen for the sake of simplicity and models that only a finite amount of energy is available to drive the temperature towards the desired (normalized) state $u\equiv 0$. We shall see that a "good design" of the thermostat crucially depends on the choice of the parameter $\beta\geqslant 0$. Our analysis suggests that the optimal design is obtained for $\beta_r\approx 0.5792$, while the choice of too large a parameter leads to oscillations. In fact there exists a critical parameter $\beta_0\approx 5.6655$ at which Hopf bifurcation from the trivial solution occurs: this is the explanation that the model provides for the oscillatory behaviour of the thermostat.

If we consider an alternative model where the boundary condition at $x=0$ is replaced by its local version

$$u_x(0)=\tanh(\beta u(0)),$$

we are in the situation of separated boundary conditions. A general result on scalar reaction-diffusion equations (cf. [2]) then shows that each solution of the corresponding evolution problem converges to a single stationary state. Therefore no periodic solutions are possible when local boundary conditions are imposed. Observe that in both of the models being considered, the only steady state (for $\beta>0$) is the trivial one. Thus in the local model the trivial solution attracts all initial data.

For the nonlocal boundary condition we show in Theorem 3.2 that the trivial solution is locally exponentially stable for $\beta\in(0,\beta_0)$. Unfortunately, we are not able to apply the stability criterion given in [9, Section 5] to show that the bifurcating periodic solution inherits the stability of the trivial solution, i.e., that the Hopf bifurcation at β_0 is supercritical. It is, however, straightforward to implement a Crank–Nicholson difference scheme, which provides strong numerical evidence in favour of the stability of the bifurcating periodic solutions. The same numerical computations even suggest that for each $\beta>\beta_0$ there is a unique stable periodic orbit.

Another interesting property of the nonlocal model is that infinitely many Hopf bifurcations seem to occur as the parameter β tends to infinity. However, we conjecture that only the first Hopf bifurcation at β_0 is supercritical and that the subsequent ones are subcritical, thus generating unstable periodic orbits.

Finally, we note that to the best of our knowledge all examples for Hopf bifurcation in reaction-diffusion equations are obtained for systems rather than for a single equation. Thus, in this sense, the evolution problem (1) is the simplest example to illustrate the phenomenon of Hopf bifurcation in the context of reaction-diffusion equations.

2. THE GLOBAL SEMIFLOW

We briefly sketch how the general results in [1] can be used to associate an abstract Cauchy problem with (1) via a suitable weak reformulation. In the following we set $H^1 := H^1(0, \pi)$ and, deviating from the standard notation, we denote its dual space $(H^1)'$ by H^{-1}. By a weak solution of (1) we mean a function $u \in C([0, \infty), H^{-1}) \cap C((0, \infty), H^1) \cap C^1((0, \infty), H^{-1})$ which satisfies

$$\begin{cases} \int_0^\pi \dot{u}\varphi \, dx + \int_0^\pi u_x \varphi_x \, dx = -\tanh(\beta u(\pi)) \, \varphi(0), \qquad t > 0, \\ u(0) = u_0, \end{cases} \tag{2}$$

for all $\varphi \in H^1$. As in [1, (8.18)] we define the unbounded operator $A\colon H^1 \subset H^{-1} \to H^{-1}$ by

$$\langle Au, \varphi \rangle = \int_0^\pi u_x \varphi_x \, dx, \qquad u, \varphi \in H^1.$$

Defining the trace operators $\gamma_0, \gamma_\pi \in \mathscr{L}(H^1, \mathbb{R})$ by $\gamma_0 u = u(0)$ and $\gamma_\pi u = u(\pi)$ we can rewrite (2) as

$$\langle \dot{u}, \varphi \rangle + \langle Au, \varphi \rangle = -\langle \tanh(\beta \gamma_\pi u), \gamma_0 \varphi \rangle. \tag{3}$$

Bu duality we are finally led to consider the abstract Cauchy problem

$$\begin{cases} \dot{u} + Au = -\gamma_0' \tanh(\beta \gamma_\pi u), \qquad t > 0, \\ u(0) = u_0, \end{cases} \tag{4}$$

in H^{-1}. This evolution problem is now amenable to semigroup methods and its solutions are in one-to-one correspondence with the weak solutions. As a consequence we obtain a continuous local semiflow (ϕ, H^1) consisting of classical solutions of the initial boundary value problem (1), as can be shown by using bootstrapping arguments. The sublinearity of the boundary conditions implies that the semiflow (ϕ, H^1) is even global.

3. THE STABILITY OF THE STATIONARY SOLUTION

We first observe that for $\beta > 0$ the trivial solution $u \equiv 0$ is the only stationary solution of (1). To study its stability properties we consider the linearization (6) of the parameter dependent problem (1) at $u = 0$. We are therefore interested in the eigenvalue problem

$$\begin{cases} -\varphi_{xx} = \lambda\varphi, & x \in (0, \pi), \\ \mathscr{B}_1 \varphi := \varphi_x(0) - \beta\varphi(\pi) = 0, & \\ \mathscr{B}_2 \varphi := \varphi_x(\pi) = 0. & \end{cases} \tag{5}$$

The principle of linearized stability (cf. e.g. [3, 5, 6]) may be applied to obtain results on the stability of the trivial solution. We note, however, that this principle is formulated in the framework of abstract parabolic equations in Banach spaces. In other words, we have to consider the linearization of (4) at $u = 0$

$$\dot{\varphi} + A\varphi = -\beta\gamma_0'\gamma_\pi\varphi, \qquad t > 0,$$

which is just the abstract formulation of

$$\begin{cases} \partial_t \varphi - \varphi_{xx} = 0, & (x, t) \in (0, \infty), \\ \mathscr{B}_1 \varphi = 0, & t \in (0, \infty), \\ \mathscr{B}_2 \varphi = 0, & t \in (0, \infty). \end{cases} \tag{6}$$

Since the spectrum $\sigma(A(\beta))$ of the operator

$$A(\beta) := A + \beta\gamma_0'\gamma_\pi, \qquad \beta \in [0, \infty),$$

is given by the eigenvalues of the problem (5) we only need to study the latter. The identity of the two spectra is a consequence of the general results on interpolation-extrapolation scales contained in [1, Section 6].

The Spectrum of the Linearization

To compute the eigenvalues of (5) we make the standard "Ansatz"

$$\varphi(x) = Ae^{i\rho x} + Be^{-i\rho x},$$

with $A, B, \rho \in \mathbb{C}$. The eigenvalues λ are then given by

$$\lambda = \rho^2,$$

for each $\rho \in \mathbb{C}$ that is a zero of the function

$$\omega(\rho, \beta) := \det \begin{bmatrix} \mathscr{B}_1 e^{i\rho x} & \mathscr{B}_2 e^{i\rho x} \\ \mathscr{B}_1 e^{-i\rho x} & \mathscr{B}_2 e^{-i\rho x} \end{bmatrix}.$$

A straightforward computation yields

$$\omega(\rho, \beta) = \rho \sin(\pi\rho) - \beta. \tag{7}$$

Obviously, complex eigenvalues of (5) are only present in complex conjugate pairs. Furthermore $\omega(\cdot, \beta)$ is an entire function for fixed β, therefore it has at most countably many isolated zeros. To study the zeros of the function ω we decompose it in its real and imaginary parts to obtain the system

$$\begin{cases} a \sin(\pi a) \cosh(\pi b) - b \cos(\pi a) \sinh(\pi b) = \beta, \\ a \cos(\pi a) \sinh(\pi b) + b \sin(\pi a) \cosh(\pi b) = 0, \end{cases} \tag{8}$$

for $\rho = a + ib$ with $a, b \in \mathbb{R}$.

Since the spectrum of (5) is known for $\beta = 0$ (Neumann spectrum) we can apply a classical result about the continuous dependence on a parameter of the roots of an entire function to discuss the changes in the spectrum of $A(\beta)$ as β increases. We collect some preliminary observations in the following

PROPOSITION 3.1. *There exists a $\beta_r > 0$ such that for each $\beta \in [0, \beta_r]$ all the zeros of $\omega(\cdot, \beta)$ are real. Furthermore for any $\beta \in [0, \infty)$ all but finitely many zeros of $\omega(\cdot, \beta)$ are real. Consequently all but finitely many eigenvalues of* (5) *are real.*

Proof. Clearly for $\beta = 0$ the solutions of $\omega(\rho, 0) = 0$ are all real and are given by

$$\rho = 0, \pm 1, \pm 2, \ldots.$$

By the continuous dependence on the parameter β of the zeros of the entire function $\omega(\cdot, \beta)$ (cf. [4, 9.17.4]) and the order conservation property, we can read off the change in the solution set of $\omega(\rho, \beta) = 0$ from the qualitative behaviour of the function $\rho \sin(\pi\rho)$ for $\rho \in \mathbb{R}$, which is plotted in Fig. 1.

By [4] we infer that all the zeros of $\omega(\cdot, \beta)$ are real until β reaches β_r, the value of the first positive maximum of $\rho \sin(\pi\rho)$. Thus

$$\beta_r \approx 0.5792.$$

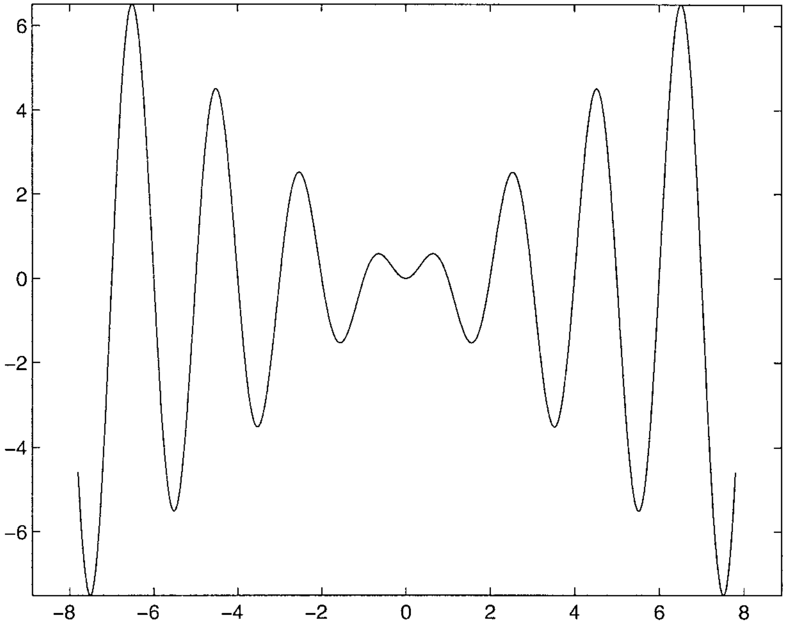

FIGURE 1

The qualitative behaviour of $\rho \sin(\pi\rho)$ also reveals that for any given $\beta \in [0, \infty]$ all but finitely many zeros of $\omega(\cdot, \beta)$ are real. ▮

The arguments in the previous proof tell us how the Neumann spectrum evolves as β increases: pairs of consecutive Neumann eigenvalues move toward each other along the real line and form a complex conjugate pair after merging. With increasing β more and more pairs of eigenvalues leave the real line.

The following theorem is the main result of this section.

THEOREM 3.2. *There exists* $\hat{\beta} > 0$ *such that the trivial solution of* (1) *is locally exponentially stable for* $\beta \in (0, \hat{\beta})$.

Proof. In the previous Proposition we have gained good control of the spectrum of $A(\beta)$ for $\beta \in [0, \beta_r]$. The assertion is a consequence of the principle of linearized stability. ▮

Remark 3.3. (a) For $\beta = 0$ the trivial solution of (1) is stable but not exponentially stable.

(b) From a numerical investigation of the evolution in β of the spectrum of $A(\beta)$ that we carry out in Remarks 4.4 it is reasonable to believe that the optimal design of the thermostat is obtained for $\beta = \beta_r$. In fact, we conjecture that

$$\beta_r = \max_{\beta \in [0, \infty)} \inf\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A(\beta))\}.$$

This would indeed imply that the analytic semigroup generated by $-A(\beta_r)$ has the fastest decay.

(c) From the proof of the previous Proposition we see that

$$0 \notin \sigma(A(\beta))$$

for $\beta > 0$. Hence the trivial solution can only loose its stability at some $\beta \geqslant \hat{\beta}$ if a pair of complex conjugate eigenvalues crosses the imaginary axis. In the next section we will show that such a crossing actually occurs and that a Hopf bifurcation takes place.

(d) We conjecture that in Theorem 3.2 the trivial solution is even globally attractive. Unfortunately the canonical "Ansatz" for a Liapunov function to (1)

$$\begin{aligned} \mathscr{L}(u) &:= \tfrac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_{\partial\Omega} G(u) \, d\sigma, \\ G(\xi) &:= \int_0^\xi \tanh(\eta) \, d\eta = \ln \cosh(\xi) \end{aligned} \tag{9}$$

seems not to be very useful in trying to prove this conjecture. This functional, however, turns out to be a good Liapunov function when the heat equation is complemented with the separated boundary conditions

$$u_x(0, t) = \tanh(\beta u(0, t)), \qquad u_x(\pi, t) = 0, \quad t \in (0, \infty). \tag{10}$$

As a consequence of the LaSalle invariance principle (cf. [8, Theorem 2.3]) or alternatively by the convergence result in [2] the trivial solution of the local version of (1) is globally attractive for each $\beta > 0$. In the next section we shall see that the nonlocal problem has a very different qualitative behaviour.

4. THE HOPF BIFURCATION

In the following proposition we determine the value of β at which a pair of nonzero complex conjugate eigenvalues of (5) crosses the imaginary axis for the first time.

PROPOSITION 4.1. *There exists $\beta_0 > 0$, $\beta_0 \approx 5.6655$, such that*

$$\sigma(A(\beta)) \cap [\mathrm{Re}(z) \leqslant 0] = \varnothing$$

for $\beta \in (0, \beta_0)$ and

$$\sigma(A(\beta_0)) \cap i\mathbb{R} = \{\pm i\omega_0\}$$

for some $\omega_0 > 0$.

Proof. We look for purely imaginary eigenvalues $\lambda = \rho^2$ of (5). Thus it suffices to look for zeros (ρ, β) of $\omega(\rho, \beta) = 0$ where ρ is of the form

$$\rho = a + ib = \alpha \sqrt{2}\, e^{i(\pi/4)},$$

with $\alpha \in \mathbb{R} \backslash \{0\}$. Inserting $a = \alpha$ and $b = \alpha$ into the system (8) we obtain the following simplified system for α and β

$$\begin{cases} h_1(\alpha) := 2\alpha \sin(\pi\alpha) \cosh(\pi\alpha) = \beta, \\ h_2(\alpha) := -2\alpha \cos(\pi\alpha) \sinh(\pi\alpha) = \beta. \end{cases} \tag{11}$$

Note that $(-\alpha, \beta)$ is a solution whenever (α, β) is one, both leading to the same eigenvalue ρ^2 of $A(\beta)$. It is now easy to study the qualitative behaviour of the two functions h_1 and h_2. In Fig. 2 the solid line is a plot of h_1 and the dashed lines are plots of the functions

$$\pm 2\alpha \cosh(\pi\alpha).$$

Analogously in Fig. 3 the solid line is a plot of h_2 and the dashed lines are plots of

$$\pm 2\alpha \sinh(\pi\alpha).$$

The behaviour of h_1 and h_2 shows that there is a smallest positive value of β that admits a solution of (11). A numerical computation yields

$$\beta_0 \approx 5.6655.$$

This common value of h_1 and h_2 must obviously be attained at some $\alpha_0 \in (\frac{1}{2}, 1)$; in fact

$$\alpha_0 \approx 0.7528. \tag{12}$$

Figure 4 shows the first intersection of h_1 and h_2. ∎

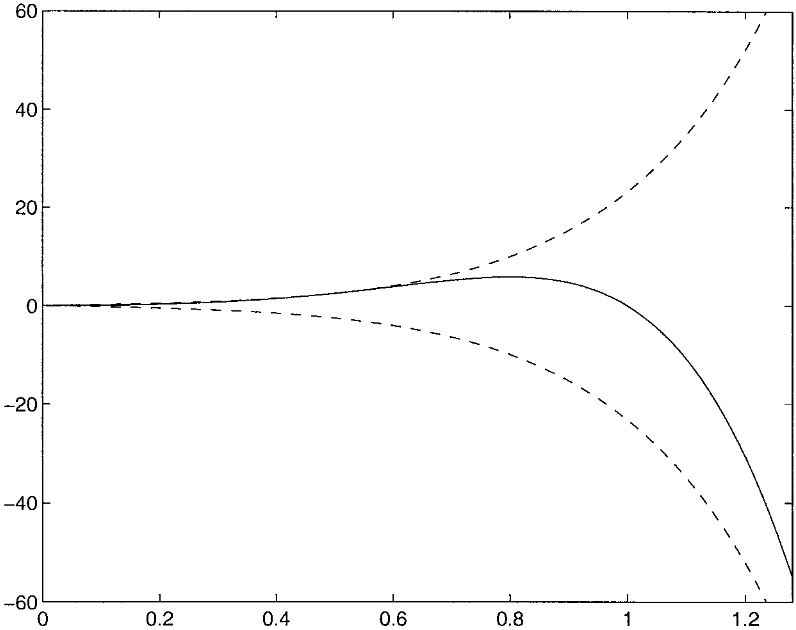

FIGURE 2

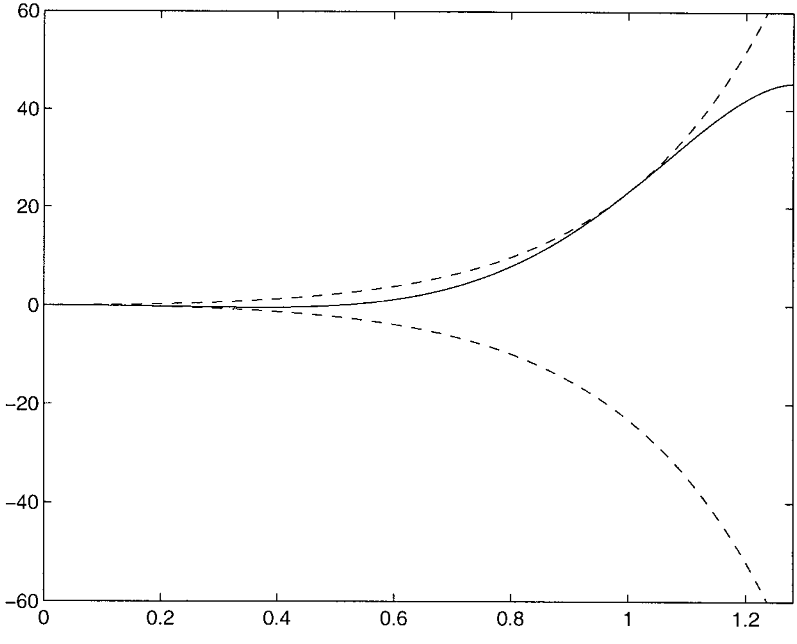

FIGURE 3

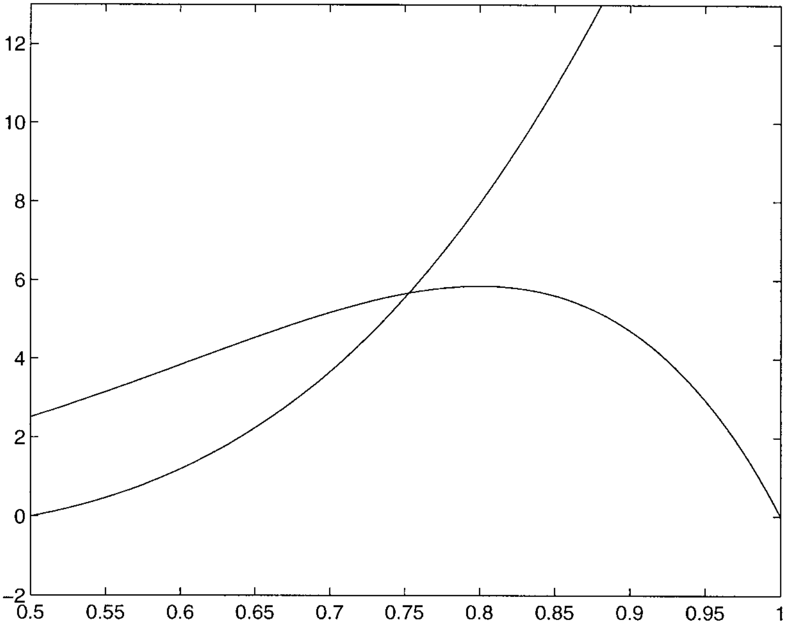

FIGURE 4

Remarks 4.2. (a) To find all solutions (α, β) with $\alpha, \beta > 0$ of the system (11) we discuss the solutions of

$$h_1(\alpha) = h_2().$$

This is clearly equivalent to solving

$$\tan(\pi\alpha) = -\tanh(\pi\alpha).$$

A plot of these functions shows the location of all solutions and the validity of the following asymptotic formula for large $k \in \mathbb{N}$

$$\alpha_k \approx \frac{1}{\pi} \arctan(-1) + k.$$

Hence there are infinitely many solutions (α_k, β_k) $(k = 0, 1, 2, ...)$ of (11) with $\alpha_k < \alpha_{k+1}$, $\beta_k < \beta_{k+1}$ and

$$i\mathbb{R} \cap \sigma(A(\beta_k)) = \{\pm i\alpha_k^2\}.$$

Summarizing, as β increases, infinitely many pairs of complex conjugate eigenvalues, coming from the right half plane, cross the imaginary axis, one pair at a time.

(b) The considerations in (a) show that β_0 equals $\hat{\beta}$, the constant appearing in Theorem 3.2.

Nondegeneracy

To apply the general results on Hopf bifurcation we need to check that the first crossing of a pair of complex conjugate eigenvalues at $\beta = \beta_0$ is nondegenerate, i.e.,

$$\frac{d}{d\beta} \operatorname{Re} \lambda(\beta_0) < 0, \tag{13}$$

where

$$\lambda \colon (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \to \mathbb{C}$$

with $\lambda(\beta_0) = \lambda_0$ is a parametrization of the crossing eigenvalue in the upper complex half plane. The arguments are clearly analogous for the complex conjugate eigenvalue. Such a C^1-parametrization exists by the implicit function theorem. In fact, a short computation shows that for $\rho \in \{\alpha \sqrt{2}\, e^{i(\pi/4)} \mid \alpha \in (\frac{1}{2}, 1)\}$ and $\beta > 0$ we have

$$\begin{aligned} \operatorname{Im} \frac{\partial}{\partial \rho} \omega(\rho, \beta) \\ = \pi\alpha \cos(\pi\alpha) \cosh(\pi\alpha) - \pi\alpha \sin(\pi\alpha) \sinh(\pi\alpha) + \cos(\pi\alpha) \sinh(\pi\alpha) < 0. \end{aligned}$$

To verify (13) note first that

$$\frac{d}{d\beta} \lambda(\beta) = \frac{d}{d\beta} (\rho^2(\beta)) = 2\rho(\beta) \frac{d}{d\beta} \rho(\beta).$$

The formula for the derivative of the implicit function gives

$$\frac{d}{d\beta} \rho(\beta_0) = -D_\rho \omega(\rho_0, \beta_0)^{-1} D_\beta \omega(\rho_0, \beta_0).$$

Clearly $D_\beta \omega \equiv -1$. To check (13) we just need to determine the sign of

$$\operatorname{Re} \rho_0 D_\rho \omega(\rho_0, \beta_0)^{-1},$$

which is easily seen to have the same sign as

$$2\pi\alpha_0 \cos(\pi\alpha_0)\cosh(\pi\alpha_0) + \sin(\pi\alpha_0)\cosh(\pi\alpha_0) + \cos(\pi\alpha_0)\sinh(\pi\alpha_0).$$

Inserting the approximate value of α_0, found in (12), we see that the last expression is negative. We are now in a position to apply general results on Hopf bifurcation (cf. [9, Theorem 4.1]).

THEOREM 4.3. *There exists* $\varepsilon > 0$ *such that for each* $\beta \in (\beta_0, \beta_0 + \varepsilon)$ *the semiflow* (ϕ, H^1) *possesses a unique periodic orbit close to the trivial equilibrium.*

Some remarks are in order.

Remarks 4.4. (a) It is easily checked that other nonlinearities having a similar qualitative behaviour as $\tanh(\cdot)$ lead to the same phenomena.

(b) To obtain a simple global picture of the evolution of the spectrum of $A(\beta)$ as β increases we have used a numerical routine to plot the zero level sets in the (a, b)-plane of the following two functions appearing in the system (8)

$$a\sin(\pi a)\cosh(\pi b) - b\cos(\pi a)\sinh(\pi b) - \beta \qquad \text{(dashed line)} \tag{14}$$

and

$$a\cos(\pi a)\sinh(\pi b) + b\sin(\pi a)\cosh(\pi b) \qquad \text{(solid line).} \tag{15}$$

Due to the symmetry of (8) we consider positive values of a only. Note that the second function is independent of β and therefore its zero level set does not move with β. To give an impression of the evolution we plot the level sets for the values $\beta = \beta_r \approx 0.5792$ in Fig. 5 and $\beta = \beta_0 \approx 5.6655$ in Fig. 6. Observe that the a and b axis are part of the zero level set of the function (15). The zeros of $\omega(\beta, \rho)$, where $\rho = a + ib$, are the intersections of the dashed and solid lines. With increasing β more and more "fingers" are formed and move away from the a axis. To track the evolution of the eigenvalues simply consider the curves $\lambda(\beta) = \rho^2(\beta)$ in the complex plane traced by the points of intersection of the dashed and solid lines.

(c) Remark 4.2 shows that infinitely many complex conjugate pairs cross the imaginary axis as β tends to infinity and numerical evidence suggests that all these crossings are indeed nondegenerate. Thus infinitely many hopf bifurcations seem to take place as β tends to infinity. However, we conjecture that the only supercritical bifurcation happens at β_0 and that all the subsequent Hopf bifurcations are subcritical and generate unstable periodic orbits.

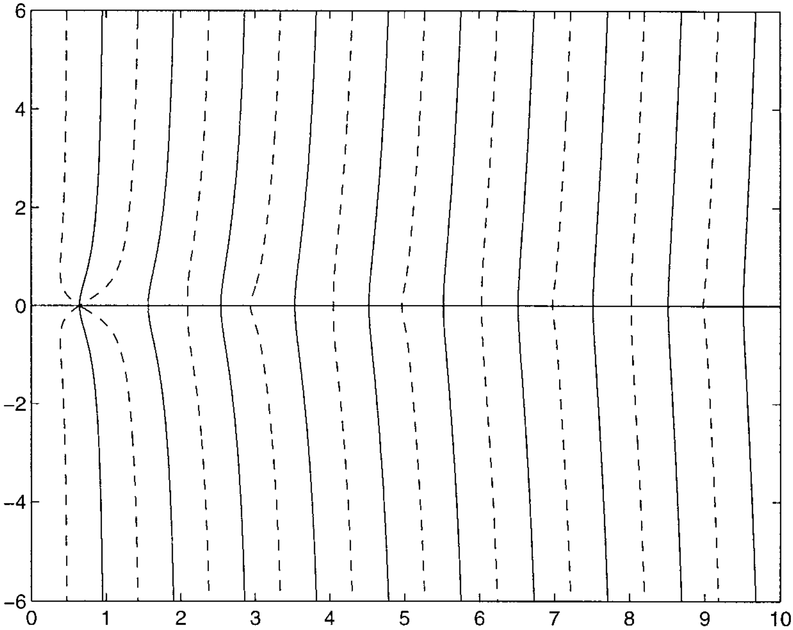

FIGURE 5

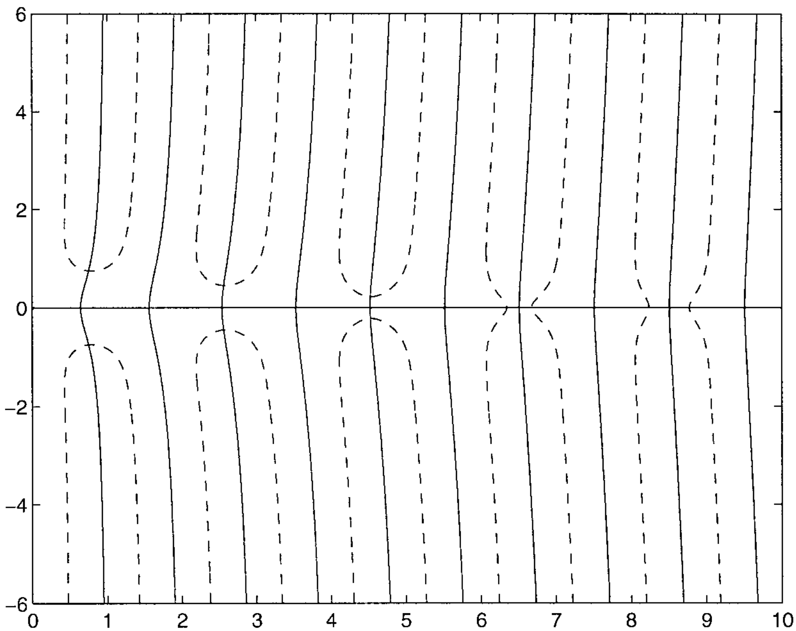

FIGURE 6

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